Linear Algebra and its Applications 484 (2015) 396–408

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Some characterizations of the trace norm triangle equality ☆

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ARTICLE INFO

Article history:
Received 31 March 2015
Accepted 8 July 2015
Available online 24 July 2015
Submitted by P. Semrl

MSC:
47B10
47B15
47A63

Keywords:
Trace-class operator
Trace norm
Triangle inequality

ABSTRACT

In this paper, we investigate some equalities for the trace-class operators on a Hilbert space. For two trace-class operators A and B, we get some equivalent conditions for $\|A\|_1 + \|B\|_1 = \|A + B\|_1$, where $\|A\|_1$ is the trace norm of the trace-class operator A. Particularly, we show that $\|A\|_1 + \|B\|_1 = \|A + B\|_1$ if and only if $|A| + |B| = |A + B|$ for two trace-class operators A and B. This condition is related to a result of Ando and Hayashi. Moreover, some characterizations of the equality $\|A\|_1 + \|A^*\|_1 = \|A + A^*\|_1$ and other relevant results are given.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and $\mathcal{B}(\mathcal{H})$ ($\mathcal{B}(\mathcal{H}, \mathcal{K})$) be the set of all bounded linear operators on $\mathcal{H}$ (from $\mathcal{H}$ to $\mathcal{K}$). For an operator $A \in \mathcal{B}(\mathcal{H})$, the adjoint of A is denoted by A^* and A is said to be self-adjoint if $A = A^*$. We also denote by $R(A)$,

☆ This work was supported by NSF of China (No. 11171197) and also supported by the Natural Science Research Plan in Shaanxi Province of China (No. 2014JM1012) and the Fundamental Research Funds for the Central Universities (GK201301007).

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http://dx.doi.org/10.1016/j.laa.2015.07.011

$\overline{R(A)}$ and $N(A)$ the range, the closed linear span of the range and the null space of A, respectively. We write $A \geq 0$ if A is a positive operator, meaning $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. As usual, $Re(A)$, $Im(A)$, and $|A|$ are the real part, the imaginary part and the absolute value of operator $A \in \mathcal{B}(\mathcal{H})$, respectively; and A^+ and A^- are the positive and negative parts of A, that is, $A^+ = \frac{A+|A|}{2}$, and $A^- = \frac{|A|-A}{2}$. For any compact operator A, let $s_1, s_2, \cdots$ be the eigenvalues of $|A|$ in decreasing order and repeated according to multiplicity. The compact operator A is said to be in the Schatten p-class C_p $(1 \leq p < \infty)$, if $\sum\limits_{i=1}^{\infty} s_i^p < \infty$. The Schatten p-norm of A is defined as $\|A\|_p = (\sum\limits_{i=1}^{\infty} s_i^p)^{\frac{1}{p}}$. This norm makes C_p into a Banach space. When $p = 1$, we get the set of all trace-class operators, denoted by $C_1(\mathcal{H})$; and when $p = 2$, C_2 is the set of Hilbert–Schmidt class.

Recently, many mathematicians have paid much attention to the Schatten p-norm and the triangle inequality of the operator value. Many interesting results have been obtained in [3,6,8–13]. Some results relevant for our purposes are the following (see [9,10]): if $\{E_i\}$ is a family of orthogonal projections satisfying $E_iE_j = 0$ then $\|A\|_p^p \geq \sum\limits_{i=1} \|E_iAE_i\|_p^p$ and for $p > 1$ equality will hold if and only if $A = \sum\limits_{i=1} E_iAE_i$. Our results (Proposition 2.5 and Remark 2.1) show that for the case of $p = 1$, the equivalent condition of $\|A\|_1 = \sum\limits_{i=1} \|E_iAE_i\|_1$ is completely different. Particularly, Ando and Hayashi have obtained that for two bounded linear operators $X, Y \in B(\mathcal{H})$, the triangle equality $|X+Y| = |X|+|Y|$ holds if and only if there exists a partial isometry U such that $X = U|X|$ and $Y = U|Y|$ in [1]. This result is also a generalization of Thompson's theorem in [15].

The purpose of this paper is to consider the properties of two trace-class operators A and B such that $\|A + B\|_1 = \|A\|_1 + \|B\|_1$. We show that this is related to the above triangle equality. Then we mainly characterize the conditions for $\|A + B\|_1 = \|A - B\|_1 = \|A\|_1 + \|B\|_1$. Our results show that this equality is equivalent to the orthogonality of the range of operators. Moreover, the structures of operator A with equation $\|A\|_1 + \|A^*\|_1 = \|A + A^*\|_1$ are given. As corollaries, we also obtain other relevant results.

2. Main results

To show our main results, the following lemmas are needed.

Lemma 2.1. *(See [5, Corollary 18.12].) Let $A \in C_1(\mathcal{H})$. If $\{e_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ are two orthonormal bases of $\mathcal{H}$, then*

$$\sum_{i=1}^{\infty} |\langle Ae_i, f_i\rangle| \leq tr(|A|) \quad \textit{and} \quad |tr(A)| \leq tr(|A|).$$

The next lemma is used in [2,4]. However, we do not find a proof. For completeness and for the convenience of readers, we provide the following proof.