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On Schatten p-norms of commutators ☆

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A R T I C L E   I N F O

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A B S T R A C T

For any odd integer $d \geq 3$, we determine the sharpest constant $C_{p,q,r}$ such that

$$\|XY - YX\|_p \leq C_{p,q,r}\|X\|_q\|Y\|_r \quad \text{for all } X, Y \in M_d,$$

where M_d denotes the set of all $d \times d$ complex matrices, $\|\cdot\|_p$, $1 \leq p \leq \infty$, denotes the Schatten p-norm on M_d, and $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} > \frac{1}{q} + \frac{1}{r}$. This is a continuation of the study of the problem considered in Wenzel and Audenaert (2010) [8].

1. Introduction

Let M_d denote the set of all $d \times d$ complex matrices and I_d denote the identity matrix of order d. For $X \in M_d$, let X^* denote the conjugate transpose of X. Let $\|\cdot\|_p$, $1 \leq p \leq \infty$, denote the Schatten p-norm on M_d. In particular, $\|\cdot\|_2$ is the Frobenius norm and $\|\cdot\|_\infty$ is the operator norm. We adopt the common usage that $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$ and $\frac{\infty}{\infty - 1} = 1$.

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In 2005, Böttcher and Wenzel [2] raised the following problem: show that an upper bound of the Frobenius norm of the commutator of any real square matrices X and Y is given by

$$\|XY - YX\|_2 \leq \sqrt{2}\, \|X\|_2 \|Y\|_2. \tag{1.1}$$

This conjecture was first proved by Vong and Jin [7] and independently by Lu [5]. Böttcher and Wenzel [3] and Audenaert [1] extended the result to complex matrices. For more results related to this conjecture, one may refer to a recent survey [4].

Wenzel and Audenaert [8] considered the following generalization: find the smallest constant $C_{p,q,r}$, $1 \leq p, q, r \leq \infty$, such that

$$\|XY - YX\|_p \leq C_{p,q,r} \|X\|_q \|Y\|_r \quad \text{for all } X, Y \in M_d. \tag{1.2}$$

In their paper, results are obtained for many instances $(p, q.r) \in [1, \infty]^3$. Nevertheless, the problem is still open in two situations:

(1) $d \geq 3$ is odd and p, q, r satisfy $\frac{1}{p} > \frac{1}{q} + \frac{1}{r}$ (except for $(p, q, r) = (1, \infty, \infty)$); and
(2) $p > 2$, $q < 2$ and $r < 2$ (except for $(p, q, r) = (\infty, 1, 1)$).

In this paper, we solve the problem for case (1). In contrast to the other (p, q, r)-regions, the constant $C_{p,q,r}$ here depends on the matrix size. Moreover, it has been observed in [8] that even and odd d will behave very different.

2. A critical case

When proving the analogous case for even d, a single rule for the region was found. However, it was conjectured that the region likely has to be split once more when d is odd. In the following lemma, the two examples suggesting the presence of such a turning point deliver a lower bound to $C_{p,q,r}$, in general.

Lemma 2.1. *Let $d \geq 3$ be an odd integer and $1 \leq p, q, r \leq \infty$. Then*

$$C_{p,q,r} \geq \max\left\{2(d-1)^{\frac{1}{p}-\frac{1}{q}-\frac{1}{r}}, 2d^{\frac{1}{p}-\frac{1}{q}-\frac{1}{r}} \cos\frac{\pi}{2d}\right\}.$$

Proof. The following two examples were used in [8] for investigating the special case $q = r = \infty$. Note that the second one is given here in an equivalent (suitably permuted) manner. Let

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [0], \quad Y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus [0].$$

Direct calculation shows that $C_{p,q,r} \geq \frac{\|XY - YX\|_p}{\|X\|_q \|Y\|_r} = 2(d-1)^{\frac{1}{p}-\frac{1}{q}-\frac{1}{r}}$.

Let $\alpha = e^{\mathbf{i}\frac{(d-1)\pi}{d}}$,