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Some perturbation results for a normalized Non-Orthogonal Joint Diagonalization problem

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A B S T R A C T

Non-Orthogonal Joint Diagonalization (NOJD) of a given real symmetric matrix set $\mathscr{A} = \{A_j\}_{j=0}^{p}$ is to find a nonsingular matrix W such that $W^\top A_j W$ for $j = 0, 1, \ldots, p$ are all as diagonal as possible. If the columns of the solution W are all required to be unit length, we call such NOJD problem as the Normalized NOJD (NNOJD) problem. In this paper, we discuss the perturbation theory for NNOJD as an optimization problem. Based on the perturbation analysis of general constrained optimization problem given in [16], we obtain an upper bound for the distance between an approximated solution of the perturbed optimal problem and the set of exact joint diagonalizers. As corollaries, a perturbation bound and an error bound are also given. Numerical examples validate the bounds.

1. Introduction

Simply speaking, the Joint Diagonalization (JD) problem can be phrased as: given a matrix set $\mathscr{A} = \{A_j\}_{j=0}^{p}$ with A_j's being real symmetric matrices of order n, find

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a nonsingular matrix $W \in \mathbb{R}^{n\times n}$ such that $W^\top A_j W$'s are all as diagonal as possible, where $W^\top$ denotes the transpose of W. Such W is often referred to as a *joint diagonalizer*. The JD problems can be divided into the Orthogonal Joint Diagonalization (OJD) problem and the Non-Orthogonal Joint Diagonalization (NOJD) problem depending on whether requiring that W is an orthogonal matrix or not. In practice, the matrices A_j's are usually constructed from empirical data, one may not be able to find a W such that $W^\top A_j W$'s are all exact diagonal. As a result, the JD problem, which may be more appropriately referred to as Approximate Joint Diagonalization (AJD) problem, is usually proposed as an optimization problem, which aims to find W to minimize certain cost functions such that the off-diagonal elements of $W^\top A_j W$'s are as close to zero as possible. Following [3], we will refer to the JD problem as Exact Joint Diagonalization (EJD) problem when exact joint diagonalization is possible, and still JD otherwise, i.e., approximation is implicitly assumed.

The JD problems arise in Blind Source Separation (BSS) and Independent Component Analysis (ICA) [9,10]. Many efforts have been devoted to solving the JD problems, and there is a long list of studies on this subject, both on the OJD problem and the NOJD problem. To name a few, for the OJD problem, the Jacobi-like method, which is achieved by successive Givens rotations (see JADE [7,8], eJADE [13] and SOBI [4]); for the NOJD problem, the alternate column diagonal center algorithm (ACDC) by Yeredor [20], the gradient descent algorithm by Joho and Mathis [12], and so on.

Although investigation for the algorithms of the JD problems has a long history, relatively little attention has been paid to the perturbation analysis of the problems. For the OJD problem, Cardoso [6] gives the first order perturbation bound of a set of commuting matrices, and then generalized by Russo in [15]. For the NOJD problem, Afsari [3] investigates the sensitivity of several cost functions using gradient flows, and gives several first order perturbation bounds, which reveals the factors that affect the sensitivity of the NOJD problem.

In this paper, we discuss the sensitivity problem of a Normalized NOJD (NNOJD) problem. To be specific, we consider the NNOJD problem as the following constrained optimization problem:

Problem 1 *(NNOJD problem).* Given a symmetric matrix set $\mathscr{A} = \{A_j\}_{j=0}^p$, find $W \in \mathcal{W}$ with

$$\begin{aligned}\mathcal{W} = \{W \in \mathbb{R}^{n\times n} \ :\ W = [\, w_1 \quad w_2 \quad \ldots \quad w_n \,] \text{ is nonsingular} \\ \text{with } \|w_i\|_2 = 1, i = 1, 2, \ldots, n\}\end{aligned} \tag{1.1}$$

such that

$$f(W) = \sum_{j=0}^{p} \mathrm{off}(W^\top A_j W) = \min, \tag{1.2}$$