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Old and new about positive definite matrices

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ABSTRACT

The first part of the paper recalls and enlarges some results which appeared in the author's paper published 50 years ago, characterizing the relationship between the diagonal entries of mutually inverse positive definite matrices. In the second part, sign patterns of positive definite matrices which are also inverse positive are studied.

1. Introduction

In [1], the author proved the following inequality containing the diagonal entries of a positive definite matrix $A = [a_{ik}]$ and its inverse $A^{-1} = [\alpha_{ik}]$.

The numbers $\sqrt{a_{ii}\alpha_{ii} - 1}$ satisfy the *polygonal inequality*, namely, each of the numbers is at most equal to the sum of the remaining. This can be written simply as

$$2 \max_i \sqrt{a_{ii}\alpha_{ii} - 1} \leq \sum_i \sqrt{a_{ii}\alpha_{ii} - 1}. \tag{1}$$

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The proof was based on the following fact, mentioned also in [3], pp. 83–86, Fact 9, and in [4]:

Observation 1.1. *If A is positive definite, then the matrix $A \circ A^{-1} - I$ is positive semidefinite and its row-sums equal to zero.*

This implies that the Hadamard product $A \circ A^{-1}$ diminished by the identity matrix is in this case a positive semidefinite matrix with row sums zero. It is thus the Gram matrix of vectors which sum to the zero vector. Consequently, their lengths satisfy the polygonal inequality.

The mentioned result led then the author to the problem to find necessary and sufficient conditions for the diagonal entries of a positive definite matrix and those of its inverse.

The first part of this paper can be considered as a continuation of the author's paper [2] published fifty years ago in which a solution of the problem was given.

For completeness, we recall the main result in a simplified form for the real case.

Theorem 1.2. *(See [2], Theorem 3.2.) Let $A = [a_{ik}]$ be an $n \times n$ positive definite matrix, $A^{-1} = [\alpha_{ik}]$ its inverse.*

Then

$$a_{ii} > 0, \quad \alpha_{ii} > 0, \quad i = 1, \ldots, n, \tag{2}$$

$$a_{ii}\alpha_{ii} \geq 1, \quad i = 1, \ldots, n, \tag{3}$$

$$2 \max_{i=1,\ldots,n} (\sqrt{a_{ii}\alpha_{ii}} - 1) \leq \sum_{i=1}^{n} (\sqrt{a_{ii}\alpha_{ii}} - 1). \tag{4}$$

Conversely, if (2), (3) and (4) are satisfied for some $2n$ numbers $a_{11}, \ldots, a_{nn}$, $\alpha_{11}, \ldots, \alpha_{nn}$, then there exists a positive definite (even real) $n \times n$ matrix with diagonal entries a_{ii} such that the diagonal entries of A^{-1} are α_{ii}.

In the first part of the present paper, we will be mainly interested in the extreme case when equality is attained in (4). This case was described in [2], specified for the real case, as follows:

Theorem 1.3. *(See [2], Theorem 3.3.) In the same notation, for an $n \times n$ matrix A, equality is attained in (4) if and only if A is diagonally congruent to the matrix*

$$P \begin{bmatrix} I_{n-1} + (\gamma - 1)cc^T & c\sqrt{\gamma^2 - 1} \\ c^T\sqrt{\gamma^2 - 1} & \gamma \end{bmatrix} P^T, \tag{5}$$

where P is some permutation matrix, c is some $(n-1)$-rowed unit real column vector and $\gamma \geq 1$ a number.