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Graphs with least eigenvalue −2: Ten years on

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The authors' monograph *Spectral Generalizations of Line Graphs* was published in 2004, following the successful use of star complements to complete the classification of graphs with least eigenvalue −2. Guided by citations of the book, we survey progress in this area over the past decade. Some new observations are included.

1. Introduction

Let G be a simple graph with n vertices. The characteristic polynomial $\det(xI - A)$ of the adjacency matrix A of G is called the *characteristic polynomial of* G and denoted by $P_G(x)$. The eigenvalues of A (i.e. the zeros of $\det(xI - A)$) and the spectrum of A (which consists of the n eigenvalues) are also called the *eigenvalues* and the *spectrum*

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of G, respectively. The eigenvalues of G are real because A is symmetric; they are usually denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, in non-increasing order. The largest eigenvalue λ_1 is called the *index* of G, and G is said to be *integral* if every eigenvalue is an integer. An overview of results on graph spectra is given in [2,5,1].

Let $\mathcal{L}$ ($\mathcal{L}^+$, $\mathcal{L}^0$) be the set of graphs whose least eigenvalue is greater than or equal to -2 (greater than -2, equal to -2). A graph is called an $\mathcal{L}$-graph ($\mathcal{L}^+$-graph, $\mathcal{L}^0$-graph) if its least eigenvalue is greater than or equal to -2 (greater than -2, equal to -2).

The *line graph* $L(H)$ of any graph H is defined as follows. The vertices of $L(H)$ are the edges of H, and two vertices of $L(H)$ are adjacent whenever the corresponding edges of H have a vertex of H in common. Interest in the study of graphs with least eigenvalue -2 began with the elementary observation that line graphs have least eigenvalue greater than or equal to -2. A natural problem arose to classify the graphs with such a remarkable property. It transpired that line graphs share this property with generalized line graphs and with some exceptional graphs (defined below).

The authors' scientific monograph on $\mathcal{L}$-graphs [4] was published in 2004, following the successful use of star complements to complete the classification of $\mathcal{L}$-graphs, and it summarized almost all results on the subject known at the time. We felt that the main problems concerning $\mathcal{L}$-graphs had been solved. However, we now see that the book did not mark the end of the story: it is the aim of this paper to review the many new and important results on star complements and spectral aspects of $\mathcal{L}$-graphs that have been obtained in the last decade.

The rest of the paper is organized as follows. Section 2 contains some technical details concerning the book and related bibliographies. In Section 3 we present some definitions and basic results required subsequently. Sections 4 and 5 contain recent results obtained by the star complement technique. In Section 6 we describe constructions for the exceptional regular $\mathcal{L}$-graphs, 1-Salem graphs and non-bipartite integral graphs with index 3. Section 7 deals with spectral characterizations and cospectral graphs. In Section 8 we discuss Hoffman graphs and limit points for the least eigenvalue of a graph in the interval $[-3, -1]$. Section 9 is concerned with the multiplicity of 0 as an eigenvalue of an $\mathcal{L}$-graph. In Section 10 we point out the relation between the signless Laplacian spectrum of a graph G and the (adjacency) spectrum of $L(G)$. Section 11 shows how results on $\mathcal{L}$-graphs have been extended to signed graphs. Section 12 deals with applications to control theory and computer science, while Section 13 summarizes developments in a miscellany of other topics.

2. Corrections, reviews and citations

Appendix A contains a list of errata in [4], taken from the website www.cs.stir.ac.uk/~pr/, where a list of corrections is currently maintained; some of these corrections were already given in [41]. The first section in our bibliography below gives a list of complete references that were not given fully in [4]. Refs. [BeSi] and [CvSt] from [4] refer to [4] itself, while the paper [144] cites [4] as a manuscript.