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Some upper bounds on the eigenvalues of uniform hypergraphs

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ABSTRACT

Let $\mathcal{H}$ be a uniform hypergraph. Let $\mathcal{A}(\mathcal{H})$ and $\mathcal{Q}(\mathcal{H})$ be the adjacency tensor and the signless Laplacian tensor of $\mathcal{H}$, respectively. In this note we prove several bounds for the spectral radius of $\mathcal{A}(\mathcal{H})$ and $\mathcal{Q}(\mathcal{H})$ in terms of the degrees of vertices of $\mathcal{H}$.

1. Introduction

We denote the set $\{1, 2, \cdots, n\}$ by $[n]$. Hypergraph is a natural generalization of ordinary graph (see [1]). A hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ on n vertices is a set of

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vertices, say $V(\mathcal{H}) = \{1, 2, \cdots, n\}$ and a set of edges, say $E(\mathcal{H}) = \{e_1, e_2, \cdots, e_m\}$, where $e_i = \{i_1, i_2, \cdots, i_l\}, i_j \in [n]$, $j = 1, 2, \cdots, l$. If $|e_i| = k$ for any $i = 1, 2, \cdots, m$, then $\mathcal{H}$ is called a k-uniform hypergraph. The degree d_i of vertex i is defined as $d_i = |\{e_j : i \in e_j \in E(\mathcal{H})\}|$. If $d_i = d$ for any vertex i of hypergraph $\mathcal{H}$, then $\mathcal{H}$ is called a d-regular hypergraph. An order k dimension n tensor $\mathcal{T} =(\mathcal{T}_{i_1 i_2 \cdots i_k}) \in \mathbb{C}^{n\times n\times\cdots\times n}$ is a multidimensional array with n^k entries, where $i_j \in [n]$ for each $j = 1, 2, \cdots, k$. To study the properties of uniform hypergraphs by algebraic methods, adjacency matrix and signless Laplacian matrix of graph are generalized to adjacency tenor and signless Laplacian tensor of uniform hypergraph.

Definition 1. (See [5,8].) Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a k-uniform hypergraph on n vertices. The adjacency tensor of $\mathcal{H}$ is defined as the k-th order n-dimensional tensor $\mathcal{A}(\mathcal{H})$ whose $(i_1 \cdots i_k)$-entry is:

$$(\mathcal{A}(\mathcal{H}))_{i_1 i_2 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & \{i_1, i_2, \cdots, i_k\} \in E(\mathcal{H}) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}(\mathcal{H})$ be a k-th order n-dimensional diagonal tensor, with its diagonal entry $\mathcal{D}_{ii\cdots i}$ being d_i, the degree of vertex i, for all $i \in [n]$. Then $\mathcal{Q}(\mathcal{H}) = \mathcal{D}(\mathcal{H}) + \mathcal{A}(\mathcal{H})$ is the signless Laplacian tensor of the hypergraph $\mathcal{H}$.

The following general product of tensors, was defined in [9] by Shao, which is a generalization of the matrix case.

Definition 2. Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times\cdots\times n_2}$ and $\mathcal{B} \in \mathbb{C}^{n_2\times n_3\times\cdots\times n_{k+1}}$ be order $m \geq 2$ and $k \geq 1$ tensors, respectively. The product $\mathcal{AB}$ is the following tensor $\mathcal{C}$ of order $(m-1)(k-1)+1$ with entries:

$$\mathcal{C}_{i\alpha_1\cdots\alpha_{m-1}} = \sum_{i_2,\cdots,i_m\in[n_2]} \mathcal{A}_{ii_2\cdots i_m}\mathcal{B}_{i_2\alpha_1}\cdots\mathcal{B}_{i_m\alpha_{m-1}}, \tag{1}$$

where $i \in [n_1], \alpha_1, \cdots, \alpha_{m-1} \in [n_3] \times \cdots \times [n_{k+1}]$.

Let $\mathcal{T}$ be an order k dimension n tensor, let $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n$ be a column vector of dimension n. Then by (1) $\mathcal{T}x$ is a vector in $\mathbb{C}^n$ whose i-th component is as the following

$$(\mathcal{T}x)_i = \sum_{i_2,\cdots,i_k=1}^{n} \mathcal{T}_{ii_2\cdots i_k} x_{i_2}\cdots x_{i_k}. \tag{2}$$

Let $x^{[k]} = (x_1^k, \cdots, x_n^k)^T$. Then (see [2,8]) a number $\lambda \in \mathbb{C}$ is called an eigenvalue of the tensor $\mathcal{T}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the following eigenequations

$$\mathcal{T}x^{k-1} = \lambda x^{[k-1]}, \tag{3}$$

and in this case, x is called an eigenvector of $\mathcal{T}$ corresponding to eigenvalue λ.