Linear Algebra and its Applications 486 (2015) 173–203

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Linear transformations that are tridiagonal with respect to the three decompositions for an LR triple

Kazumasa Nomura

Tokyo Medical and Dental University, Kohnodai, Ichikawa, 272-0827 Japan

ARTICLE INFO

Article history:
Received 7 August 2015
Accepted 19 August 2015
Available online 27 August 2015
Submitted by R. Brualdi

MSC:
17B37
15A21

Keywords:
Lowering map
Raising map
Quantum group
Quantum algebra
Lie algebra

ABSTRACT

Recently, Paul Terwilliger introduced the notion of a lowering–raising (or LR) triple, and classified the LR triples. An LR triple is defined as follows. Fix an integer $d \geq 0$, a field $\mathbb{F}$, and a vector space V over $\mathbb{F}$ with dimension $d+1$. By a decomposition of V we mean a sequence $\{V_i\}_{i=0}^d$ of 1-dimensional subspaces of V whose sum is V. For a linear transformation A from V to V, we say A lowers $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i-1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$. We say A raises $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i+1}$ for $0 \leq i \leq d$, where $V_{d+1} = 0$. An ordered pair of linear transformations A, B from V to V is called LR whenever there exists a decomposition $\{V_i\}_{i=0}^d$ of V that is lowered by A and raised by B. In this case the decomposition $\{V_i\}_{i=0}^d$ is uniquely determined by A, B; we call it the (A, B)-decomposition of V. Consider a 3-tuple of linear transformations A, B, C from V to V such that any two of A, B, C form an LR pair on V. Such a 3-tuple is called an LR triple on V. Let α, β, γ be nonzero scalars in $\mathbb{F}$. The triple αA, βB, γC is an LR triple on V, said to be associated to A, B, C. Let $\{V_i\}_{i=0}^d$ be a decomposition of V and let X be a linear transformation from V to V. We say X is tridiagonal with respect to $\{V_i\}_{i=0}^d$ whenever $XV_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$. Let $\mathcal{X}$ be the vector space over $\mathbb{F}$ consisting of the linear transformations from V to V that are tridiagonal with respect to the (A, B) and (B, C) and (C, A) decompositions of V. There is a special class of LR triples, called q-Weyl type.

E-mail address: knomura@pop11.odn.ne.jp.

http://dx.doi.org/10.1016/j.laa.2015.08.016

In the present paper, we find a basis of $\mathcal{X}$ for each LR triple that is not associated to an LR triple of q-Weyl type.

1. Introduction

The equitable presentation for the quantum algebra $U_q(\mathfrak{sl}_2)$ was introduced in [3] and further investigated in [4,5]. For the lie algebra $\mathfrak{sl}_2$, the equitable presentation was introduced in [2] and comprehensively studied in [1]. From the equitable point of view, consider a finite-dimensional irreducible module for $U_q(\mathfrak{sl}_2)$ or $\mathfrak{sl}_2$. In [1,4] three nilpotent linear transformations of the module are encountered, with each transformation acting as a lowering map and raising map in multiple ways. In order to describe this situation more precisely, Paul Terwilliger introduced the notion of a lowering–raising (or LR) triple of linear transformations, and gave their complete classification (see [6]).

There are three decompositions associated with an LR triple. In the present paper, we investigate the linear transformations that act in a tridiagonal manner on each of these three decompositions. In this section, we first recall the notion of an LR triple, and then state our main results.

Throughout the paper, fix an integer $d \geq 0$, a field $\mathbb{F}$, and a vector space V over $\mathbb{F}$ with dimension $d+1$. Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra consisting of the $\mathbb{F}$-linear transformations from V to V, and let $\mathrm{Mat}_{d+1}(\mathbb{F})$ denote the $\mathbb{F}$-algebra consisting of the $(d+1) \times (d+1)$ matrices that have all entries in $\mathbb{F}$. We index the rows and columns by $0, 1, \ldots, d$.

By a *decomposition* of V we mean a sequence $\{V_i\}_{i=0}^d$ of 1-dimensional subspaces of V such that $V = \sum_{i=0}^d V_i$ (direct sum). Let $\{V_i\}_{i=0}^d$ be a decomposition of V. For notational convenience define $V_{-1} = 0$ and $V_{d+1} = 0$. For $A \in \mathrm{End}(V)$, we say A *lowers* $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i-1}$ for $0 \leq i \leq d$. We say A *raises* $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i+1}$ for $0 \leq i \leq d$. An ordered pair A, B of elements in $\mathrm{End}(V)$ is called *LR* whenever there exists a decomposition of V that is lowered by A and raised by B. In this case the decomposition $\{V_i\}_{i=0}^d$ is uniquely determined by A, B (see [6, Section 3]); we call it the (A,B)-*decomposition* of V. For $0 \leq i \leq d$ define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $0 \leq j \leq d$, $j \neq i$, where I denotes the identity in $\mathrm{End}(V)$. We have $E_i E_j = \delta_{i,j} E_i$ for $0 \leq i, j \leq d$ and $I = \sum_{i=0}^d E_i$. We call $\{E_i\}_{i=0}^d$ the *idempotent sequence* for A, B (or $\{V_i\}_{i=0}^d$).

A 3-tuple A, B, C of elements in $\mathrm{End}(V)$ is called an *LR triple* whenever any two of A, B, C form an LR pair on V. We say A, B, C is *over* $\mathbb{F}$. We call d the *diameter* of A, B, C.

Let A, B, C be an LR triple on V and let A', B', C' be an LR triple on a vector space V' over $\mathbb{F}$ with dimension $d+1$. By an *isomorphism of LR triples* from A, B, C to A', B', C' we mean an $\mathbb{F}$-linear bijection $\sigma : V \to V'$ such that $\sigma A = A'\sigma$, $\sigma B = B'\sigma$,