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Flat phenomena of 2-variable weighted shifts

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We study the class of mono-weakly 2-hyponormal 2-variable weighted shifts $\mathbf{T} \equiv (T_1, T_2)$ (resp. hyponormal with quadratically hyponormal T_1 and T_2) with two consecutive equal weights in the weight sequence of each of the coordinate operators. We show that under natural assumptions on the coordinate operators, the presence of consecutive equal weights in T_1 and T_2 leads to the flatness, in a way that resembles the situation for 1-variable weighted shifts.

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1. Introduction

For $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha \equiv \text{shift}\ (\alpha_0, \alpha_1, \cdots) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_k := \alpha_k e_{k+1}$ (all $k \geq 0$), where $\{e_k\}_{k=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. If $\alpha_{k+1} = \alpha_k$ for all $k \geq 1$, W_α is then called *flat*. The flatness of weighted shifts is largely studied in the literature [3,6–10,15,16,19–22,25,27,29], particularly its results and techniques are important in the theory of subnormal operators which were introduced by Paul R. Halmos in 1950 [24] for the purpose of the study of dilations and extensions of operators on a Hilbert space. In [16,21], the authors first introduced the flatness for a commuting hyponormal 2-variable weighted shift (T_1, T_2) which is the correct analogue of the flatness for a single weighted shift W_α.

We denote the class of commuting pairs of operators on Hilbert space by $\mathfrak{C}_0$, the class of commuting pairs of subnormal operators on Hilbert space by $\mathfrak{H}_0$, the class of subnormal pairs by $\mathfrak{H}_\infty$, and for an integer $k \geq 1$, the class of k-hyponormal pairs in $\mathfrak{H}_0$ by $\mathfrak{H}_k$. Clearly, we have $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$. In [14], the authors provided a multivariable analogue of the Bram–Halmos criterion [2,4] for subnormality and a matricial characterization of k-hyponormality for multivariable weighted shifts. As an application of the main result in [14], the authors built an example which exhibits the gap between (joint) k-hyponormality and $(k+1)$-hyponormality for each $k \geq 1$. However, we still don't know whether there exists a gap between (jointly) weak k-hyponormality and weak $(k+1)$-hyponormality in $\mathfrak{C}_0$ for each $k \geq 1$. We say that a pair $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{C}_0$ is *weakly k-hyponormal* (resp. *polynomially hyponormal* when $k = \infty$) if $(p_1(T_1, T_2), p_2(T_1, T_2))$ is hyponormal for all polynomials $p_1,\ p_2 \in \mathbb{C}[z, w]$ with $\deg p_1,\ \deg p_2 \leq k$ (cf. [22]). From the definition of the weakly k-hyponormal, it seems highly nontrivial to check the weak k-hyponormality of $\mathbf{T} \in \mathfrak{C}_0$, even if $k = 2$. We thus introduce the notion of mono-weak k-hyponormality for each $k \geq 1$. We say that $(T_1, T_2) \in \mathfrak{C}_0$ is *mono-weakly k-hyponormal* (resp. *mono-polynomially hyponormal* when $k = \infty$) if $p(T_1, T_2)$ is hyponormal for all polynomials $p \in \mathbb{C}[z, w]$ with $\deg p \leq k$. For $S, T \in \mathcal{B}(\mathcal{H})$, we let $[S, T] := ST - TS$ and $M_k(T) := \left([T^{j*}, T^i]\right)_{i,j=1}^{k}$. We recall that $T \in \mathcal{B}(\mathcal{H})$ is *weakly k-hyponormal* for each $k \geq 1$ if for all $a_1, \cdots, a_k \in \mathbb{C}$ and $x \in \mathcal{H}$

$$\Big\langle M_k(T)\, f(k, x),\, f(k, x) \Big\rangle \geq 0, \tag{1}$$

where $f(k, x) := (a_1 x, \cdots, a_k x)^t$. For $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{C}_0$, we let

$$M_k(T_1, T_2) := \left([(T_2^q T_1^p)^*, T_2^n T_1^m]\right)_{\substack{1 \leq m+n \leq k \\ 1 \leq p+q \leq k}}.$$

Based on (1) and Proposition 5.1 given below, we can see that a pair $\mathbf{T} \in \mathfrak{C}_0$ is mono-weakly k-hyponormal (resp. mono-polynomially hyponormal when $k = \infty$) if and only