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Conservative algebras of 2-dimensional algebras

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ABSTRACT

In 1990 Kantor introduced the conservative algebra $W(n)$ of all algebras (i.e. bilinear maps) on the n-dimensional vector space. In case $n > 1$ the algebra $W(n)$ does not belong to well-known classes of algebras (such as associative, Lie, Jordan, Leibniz algebras). We describe the algebra of all derivations of $W(2)$ and subalgebras of $W(2)$ of codimension one. We also study similar problems for the algebra W_2 of all commutative algebras on the two-dimensional vector space and the algebra S_2 of all commutative algebras with trace zero multiplication on the two-dimensional space.

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1. Introduction

We work over an arbitrary field $\mathbb{F}$ of characteristic zero. An algebra may not have a unity and may not be associative. By a multiplication on a vector space W we mean a bilinear mapping $W \times W \to W$. Given a vector space W, a linear mapping A on W, and a bilinear mapping $B : W \times W \to W$, we can define a multiplication $[A, B] : W \times W \to W$ as follows:

$$[A, B](x, y) = A(B(x, y)) - B(A(x), y) - B(x, A(y))$$

for all $x, y \in W$. For an algebra A with multiplication P and $x \in A$ we write L_x^P for the left multiplication on x. If the multiplication P is fixed, then we write L_x instead of L_x^P.

In 1990 Kantor [15] defined the multiplication $\cdot$ on the set of all algebras (i.e. all multiplications) on the n-dimensional vector space V_n as follows:

$$A \cdot B = [L_e^A, B],$$

where A and B are multiplications and e is the fixed vector from V_n. In case $n > 1$ the resulting algebra $W(n)$ does not belong to the well-known classes of algebras (such as associative, Lie, Jordan, Leibniz algebras). The algebra $W(n)$ turns out to be a conservative algebra (see below).

In 1972 Kantor [11] introduced conservative algebras as a generalization of Jordan algebras. Namely, an algebra A with the multiplication $P(x, y) = xy$ and the underlying vector space W is a *conservative algebra* if a new multiplication F can be defined on the vector space W in such a way that

$$[L_b, [L_a, P]] = -[L_{F(a,b)}, P] \tag{1}$$

for all $a, b \in W$. In other words, the following identity holds for all a, b, x, y from W:

$$\begin{aligned} b(a(xy) - (ax)y - x(ay)) - a((bx)y) + (a(bx))y \\ + (bx)(ay) - a(x(by)) + (ax)(by) + x(a(by)) \\ = -F(a, b)(xy) + (F(a, b)x)y + x(F(a, b)y). \end{aligned} \tag{2}$$

The algebra with the multiplication F is said to be *associated* to A.

As an example, it is easy to see that every 4-nilpotent algebra is a conservative algebra with $F(a, b) = 0$.

The algebra $W(n)$ plays a similar role in the theory of conservative algebras as the Lie algebra of all $n \times n$ matrices gl_n plays in the theory of Lie algebras. Namely, in [15] Kantor considered the category $\mathcal{S}_n$ whose objects were conservative algebras of non-Jacobi dimension n. It was proven that the algebra $W(n)$ is the universal attracting object in this category, i.e., for every algebra M of $\mathcal{S}_n$ there exists a canonical homomorphism