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Nilpotent evolution algebras over arbitrary fields

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A R T I C L E I N F O

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A B S T R A C T

The paper is devoted to the study of annihilator extensions of evolution algebras and suggests an approach to classify finite-dimensional nilpotent evolution algebras. Subsequently nilpotent evolution algebras of dimension up to four are classified.

1. Introduction

Evolution algebras were introduced in 2006 by J.P. Tian and P. Vojtechovsky in their paper "Mathematical concepts of evolution algebras in non-Mendelian genetics" (see [10]). Later on, Tian laid the foundations of evolution algebras in his monograph [11]. These algebras present many connections with other mathematical fields including graph theory, group theory, Markov chains, dynamical systems, knot theory, 3-manifolds and the study of the Riemann-Zeta function (see [11]).

Evolution algebras are in general non-associative and do not belong to any of the well-known classes of non-associative algebras such as Lie algebras, alternative algebras,

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or Jordan algebras. Therefore, the research on these algebras follows different paths (see [2–4,8,12]).

A classical problem in study of any class of algebras is to know how many different (up to isomorphism) algebras exist for each dimension. In this way in [1,6,7,9], the classifications of nilpotent Jordan algebras, nilpotent Lie superalgebras, nilpotent associative algebras and nilpotent Lie algebras of low dimensions were given.

In this paper we study the class of nilpotent evolution algebras. Our aim is to describe a method for classifying nilpotent evolution algebras. In [3], the equivalence between nilpotent evolution algebras and evolution algebras which are defined by upper triangular matrices is proved. In [11], J. Tian defined an evolution algebra associated to any directed graph. In [5], A. Elduque and A. Labra considered the reverse direction, a directed graph is attached to any evolution algebra and they proved that nilpotency of an evolution algebra is equivalent to the nonexistence of oriented cycles in the attached directed graph.

The paper is organized as follows. In Section 2, we give some basic concepts about evolution algebras. Section 3 is devoted to the description of construction of evolution algebras with non-trivial annihilator as annihilator extensions of evolution algebras of lower dimensions. In Section 4, we describe a method for classifying nilpotent evolution algebras. In Section 5, we classify nilpotent evolution algebras of dimension up to three over arbitrary fields. In Section 6, four-dimensional nilpotent evolution algebras are classified over an algebraic closed field of characteristic not 2 (the charecterstic 2 case will be posted on arxiv) and over $\mathbb{R}$.

2. Preliminaries

Definition 2.1. (See [11].) An *evolution algebra* is an algebra $\mathcal{E}$ containing a basis (as a vector space) $B = \{e_1, \ldots, e_n\}$ such that $e_i e_j = 0$ for any $1 \leq i < j \leq n$. A basis with this property is called a *natural basis.*

Given a natural basis $B = \{e_1, \ldots, e_n\}$ of an evolution algebra $\mathcal{E}$,

$$e_i^2 = \sum_{j=1}^{n} \alpha_{ij} e_j$$

for some scalars $\alpha_{ij} \in \mathbb{F}$, $1 \leq i, j \leq n$. The matrix $A = \left(\alpha_{ij}\right)$ is the *matrix of structural constants* of the evolution algebra $\mathcal{E}$, relative to the natural basis B.

Definition 2.2. An *ideal* $\mathcal{I}$ of an evolution algebra $\mathcal{E}$ is an evolution algebra satisfying $\mathcal{E}\mathcal{I} \subseteq \mathcal{I}$.

Given an evolution algebra $\mathcal{E}$, consider its *annihilator*

$$ann\left(\mathcal{E}\right) := \left\{x \in \mathcal{E} : x\mathcal{E} = 0\right\}.$$