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Representations of ∗-semigroups associated to invariant kernels with values adjointable operators

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A B S T R A C T

We consider positive semidefinite kernels valued in the ∗-algebra of adjointable operators on a VE-space (Vector Euclidean space) and that are invariant under actions of ∗-semigroups. A rather general dilation theorem is stated and proved: for these kind of kernels, representations of the ∗-semigroup on either the VE-spaces of linearisation of the kernels or on their reproducing kernel VE-spaces are obtainable. We point out the reproducing kernel fabric of dilation theory and we show that the general theorem unifies many dilation results at the non-topological level.

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0. Introduction

Starting with the celebrated Naimark's dilation theorems in [22] and [23], a powerful dilation theory for operator valued maps was obtained through results of B. Sz.-Nagy [32], W.F. Stinespring [30], and their generalisations to VH-spaces (Vector Hilbert spaces) by R.M. Loynes [17], or to Hilbert C^*-modules by G.G. Kasparov [15]. Taking into account the diversity of dilation theorems for operator valued maps, there is a natural question, whether one can unify all, or the most, of these dilation theorems, under one theorem. An attempt to approach this question was made in [11] by using the notion of VH-space over an admissible space, introduced by R.M. Loynes [17,18]. As a second step in this enterprise, an investigation at the "ground level", that is, a non-topological approach, makes perfect sense. In addition, an impetus to pursue this way was given to us by the recent investigation on closely related problems, e.g. non-topological theory for operator spaces and operator systems, cf. [27,5,26,6].

The aim of this article is to present a general non-topological approach to dilation theory based on positive semidefinite kernels that are invariant under actions of $*$-semigroups and with values adjointable operators on VE-spaces (Vector Euclidean spaces) over ordered $*$-spaces. More precisely, we show that at the level of conjunction of order with $*$-spaces or $*$-algebras and operator valued maps, one can obtain a reasonable dilation theory that contains the fabric of most of the more or less topological versions of classical dilation theorems. In addition, we integrate into non-topological dilation theory, on equal foot, the reproducing kernel technique and show that almost each dilation theorem is equivalent to a realisation as a reproducing kernel space with additional properties. Our approach is based on ideas already present under different topological versions of dilation theorems in [24,7,17,3,4,21,8–10,31,12,11] and, probably, many others.

We briefly describe the contents of this article. In Section 1 we fix some terminology and facts on ordered $*$-spaces, ordered $*$-algebras, VE-spaces over ordered $*$-spaces, and VE-modules over ordered $*$-algebras. On these basic objects, one can build the ordered $*$-algebras of adjointable operators on VE-spaces or VE-modules. We provide many examples that illustrate the richness of this theory, even at the non-topological level.

Then, in Section 2, we consider the main object of investigation which refers to positive semidefinite kernels with values adjointable operators on VE-spaces. We make a preparation by showing that, although analogs of Schwarz Inequality is missing at this level of generality, some basic results can be obtained by different techniques. In order to achieve a sufficient generality that allows to recover known dilation theorems for both $*$-semigroups (B. Sz.-Nagy) and $*$-algebras (Stinespring), in view of [11], we consider positive semidefinite kernels that are invariant under actions of $*$-semigroups and that have values adjointable operators on VE-spaces. In Lemma 2.1, we show that, for a 2-positive kernel, if boundedness in the sense of Loynes is assumed for all the operators on the diagonal, then the entire kernel is made up by bounded operators. In this way we explain how the investigation of this article is situated with respect to that in [11]. We briefly show the connection between linearisations and reproducing kernel spaces at this