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Max-plus singular values

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ABSTRACT

In this paper we prove a new characterization of the max-plus singular values of a max-plus matrix, as the max-plus eigenvalues of an associated max-plus matrix pencil. This new characterization allows us to compute max-plus singular values quickly and accurately. As well as capturing the asymptotic behavior of the singular values of classical matrices whose entries are exponentially parameterized we show experimentally that max-plus singular values give order of magnitude approximations to the classical singular values of parameter independent classical matrices.

We also discuss Hungarian scaling, which is a diagonal scaling strategy for preprocessing classical linear systems. We show that Hungarian scaling can dramatically reduce the 2-norm condition number and that this action can be explained using our new theory for max-plus singular values.

0. Introduction

Max-plus algebra concerns the semiring $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with addition and multiplication operations

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$$a \oplus b = \max\{a, b\}, \quad a \otimes b = a + b, \quad a, b \in \mathbb{R}_{\max}.$$

More generally tropical algebra is the study of any semiring in which the addition operation is max or min, for example max-times, min–max and max-average.

Max-plus algebra naturally describes certain dynamical systems and operations research problems [1,2]. Max-plus algebra can also be used to approximate or bound the solutions to certain classical algebra problems, which is the topic of this paper.

An $n \times m$ max-plus matrix $G \in \mathbb{R}_{\max}^{n\times m}$ is simply an $n \times m$ array of entries from $\mathbb{R}_{\max}$. The max-plus Singular Value Decomposition (SVD) of a max-plus matrix was introduced by De Schutter and De Moor in [3]. They work in the symmetrized max-plus algebra, which is an extension of the max-plus semi-ring including a max-plus analogue of the subtraction operation. In this setting equalities are replaced with weaker relations, which they call balances. Their main result is proving the existence of a max-plus SVD which looks exactly as the classical SVD but with max replacing sum, sum replacing times and balancing replacing equality. The max-plus SVD is useful for analyzing certain max-plus linear systems. De Schutter and De Moore also use the decomposition to introduce a definition of the rank of a max-plus matrix, which is useful in max-plus linear signal processing problems. However they do not provide a polynomial time algorithm for computing the max-plus SVD of a max-plus matrix $G \in \mathbb{R}_{\max}^{n\times n}$ and the method that they describe requires one to solve a difficult classical algebra problem, namely to find the asymptotic behavior of the analytic SVD of a matrix whose entries are exponentials with exponents given by the entries of the max-plus matrix, $A(t) = (a_{ij}(t))$ with

$$a_{ij}(t) = b_{ij} \exp(g_{ij} t),$$

for generic $B = (b_{ij}) \in \mathbb{C}^{n\times n}$. In this paper we take the opposite approach! We want to use the max-plus singular values of $G = (g_{ij})$ to tell us something about the classical singular values of A, rather than the other way around. As well as enabling us to compute the asymptotics of the singular values of a matrix whose entries are exponentials, we show that max-plus singular values can be used to approximate the log of classical singular values of a fixed matrix $M \in \mathbb{C}^{n\times m}$. The theory we develop also explains the action of Hungarian scaling, which is a diagonal-scaling/balancing technique for classical linear systems.

Using our new characterization, the max-plus singular values of an $n \times m$ max-plus matrix G can be computed in a numerically stable way with $O(k\tau)$ complexity, where $k = \min\{n, m\}$ and τ is the number of non-zero elements in the matrix. We perform these computations using our own algorithm, which is loosely based on the max-plus eigensolver algorithm of Gassner and Klinz [4]. In this paper we focus on computing the max-plus singular values rather than the max-plus SVD decomposition, but it is possible to use our results to compute the singular vectors in polynomial time using our matrix pencil description of the problem, the max-plus eigensolver algorithm and through repeated use of the max-plus algebra of pairs Cramer's rule [5, Chapter 3.5].