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On the interlacing inequalities for invariant factors

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E.M. Sá and R.C. Thompson proved that the invariant factors of a matrix over a commutative principal ideal domain and the invariant factors of its submatrices are related by a set of divisibility inequalities, called the interlacing inequalities for invariant factors. We extend this result to matrices over elementary divisor duo rings.

1. Introduction and main results

Throughout this paper, R denotes an associative ring with identity. We say that $a \in R$ is a total divisor (respectively, right divisor; left divisor) of $b \in R$ if $RbR \subseteq aR \cap Ra$

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(respectively, $Rb \subseteq Ra$; $bR \subseteq aR$). If $a, b \in R$, $a \mid b$ means that a is a total divisor of b; $a \sim b$ means that $a \mid b \mid a$, that is, $RaR = aR = Ra = RbR = bR = Rb$; hence, $a \mid a$ if and only if $aR = Ra$. Note that an element of R may not divide totally itself. We shall say that $d \in R$ is a *greatest common divisor* (gcd) of $X \subseteq R$ if d divides totally the elements of X and, for every $e \in R$ that divides totally the elements of X, $e \mid d$. In particular, if d is a gcd of X, then $d \mid d$.

We say that R is a *duo ring* if every right ideal and every left ideal of R is bilateral. It is easy to see that R is duo if and only if, for every $a \in R$, $a \mid a$. If R is a duo ring and $a, b \in R$, then a is a total divisor of b if and only if a is a right divisor of b if and only if a is a left divisor of b.

Two matrices $B, B' \in R^{m\times n}$ are said to be *equivalent* if there exist invertible matrices $X \in R^{m\times m}$ and $Y \in R^{n\times n}$ such that $B' = XBY$. We call *diagonal reduction* of $B \in R^{m\times n}$ to any matrix equivalent to B with the form

$$\begin{bmatrix} \operatorname{diag}(b_1, \dots, b_r) & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where $r \in \{0, \dots, \min\{n, m\}\}$, $b_1, \dots, b_r \in R \setminus \{0\}$ and $b_1 \mid \cdots \mid b_r$. When $B \in R^{m\times n}$ is equivalent to a diagonal reduction (1), we say that $b_1, \dots, b_r$ is a sequence of invariant factors of B. We also say that (1) is a diagonal reduction of B with invariant factors $b_1, \dots, b_r$. Note that the number r is the same in any diagonal reduction of B. For notational convenience, we add an infinite string of zeros $b_{r+1} = b_{r+2} = \cdots = 0$, which we also call invariant factors of B. In general, invariant factors are not unique.

We say that R is a *left Hermite ring* [2] if every 2×1 matrix has a diagonal reduction. If R is a left Hermite ring, then, for every matrix $B \in R^{m\times n}$, there exists an invertible matrix $X \in R^{m\times m}$ such that XB is upper triangular with the diagonal starting in the upper left corner (respectively, lower triangular with the diagonal starting in the lower right corner). Right Hermite rings are defined analogously.

We say that R is an *elementary divisor ring* [2] if every matrix over R has a diagonal reduction.

We say that R is a *valuation ring* [2] if, for every $a, b \in R$, either $a \mid b$ or $b \mid a$. Valuation rings are duo rings. Valuation rings are also elementary divisor rings [2].

We say that R is a *domain* if $R \neq \{0\}$ and R has no zero divisors. We say that a domain R is a *principal ideal domain* if every right ideal and every left ideal of R is principal. Principal ideal domains are elementary divisor rings [1]. Other conditions for a ring to be an elementary divisor ring are known. See [6] for a survey.

Our main objective is to prove Theorems 1 and 4 below. Proofs will be given later. These theorems were proved in [4,5] when R is a commutative principal ideal domain. In [3], it was observed that Theorem 1 is valid when R is a commutative elementary divisor domain. A particular case of Theorem 1 also appears in [2] when R is a valuation ring.