Linear Algebra and its Applications 486 (2015) 484–493

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Matrix form of the inverse Young inequalities

S.M. Manjegani *, A. Norouzi

Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, 84156-83111, Iran

ARTICLE INFO

Article history:
Received 22 May 2015
Accepted 18 August 2015
Available online 10 September 2015
Submitted by P. Semrl

MSC:
15A42
46L05

Keywords:
Inverse Young inequality
Positive semi-definite matrix
Singular values

ABSTRACT

We use operator monotone and operator convex functions to prove an inverse to the Young inequality for eigenvalues of positive definite matrices and then apply it to obtain a matrix inverse Young inequality which can be considered as a complement of a result of T. Ando. Also, we give a necessary and sufficient condition for the equality.

1. Introduction

Some of the most important inequalities, as well as some equalities in complex numbers admit generalisations in a matrix context. The equality $|z\overline{w}| = |z|\,|w|$, the triangle inequality $|z+w| \leq |z| + |w|$ and the arithmetic mean-geometric inequality $|z\overline{w}| \leq \frac{1}{2}(|z|^2 + |w|^2)$, are all in evidence. (See [10] and [4] for generalisations of the second and third to complex matrices.)

* Corresponding author.
E-mail addresses: manjgani@cc.iut.ac.ir (S.M. Manjegani), asghar.norouzi@math.iut.ac.ir (A. Norouzi).

http://dx.doi.org/10.1016/j.laa.2015.08.022
0024-3795/

Another such inequality is the Young inequality:

$$|z\overline{w}| \leq \frac{1}{p}|z|^p + \frac{1}{q}|w|^q \tag{1}$$

in which $p, q \in (1, \infty)$ are conjugate exponents. Moreover, equality holds if and only if $|z|^p = |w|^q$.

In what follows, $M_n(C)$ denotes the set (the C^*-algebra) of all $n \times n$ complex matrices. A Hermitian matrix $A \in M_n(C)$ is called positive semi-definite (resp. positive definite) if $\langle Ax, x\rangle \geq 0$ (resp. $\langle Ax, x\rangle > 0$) for each $x \in C^n$. The set $M_n^+(C)$ of all positive semi-definite matrices is a closed convex cone in $M_n(C)$ and makes the set of all Hermitian matrices partially ordered: for Hermitian matrices A and B, $A \leq B$ if and only if $B - A \in M_n^+(C)$.

For a Hermitian matrix A we arrange the eigenvalues of A in non-increasing order as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. For $X \in M_n(C)$, $|X|$ stands for the unique positive square root of X^*X. Eigenvalues of $|X|$ are called singular values of X and are also arranged in non-increasing order as $s_1(X) \geq s_2(X) \geq \cdots \geq s_n(X)$.

The Young inequality (1) was extended to complex matrices in the special case $p = q = 2$ by Bhatia and Kittaneh in [7] and in the general case by Ando in [1] as follows:

Theorem 1.1. *For each pair of complex matrices A and B in $M_n(C)$ and each pair of conjugate exponents p and q in $(1, \infty)$*

$$\lambda_j(|AB^*|) \leq \lambda_j\Big(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\Big), \quad j = 1, 2, \cdots, n. \tag{2}$$

Equivalently, there exists a unitary matrix U such that

$$U^*|AB^*|U \leq \frac{1}{p}|A|^p + \frac{1}{q}|B|^q. \tag{3}$$

In [9], Hirzallah and Kittaneh proved a refinement of the Young inequality for l^2 norms of matrices. The original version of the inequality was given which was by Bhatia and Parthasarathy in [5]. Hirzallah and Kittaneh showed that equality holds in (2) if and only if $|A|^p = |B|^q$. Generalisation of (3) to compact operators acting on a complex separable Hilbert space was established by Erlijman, Farenick and Zeng [8]. Later, Argerami and Farenick established the case of equality for trace class operators in [2].

Note that Young's inequality (1) can also be written in the form

$$|z\overline{w}| \leq \nu\,|z|^{\frac{1}{\nu}} + (1-\nu)|w|^{\frac{1}{1-\nu}}, \;\; \nu \in (0, 1). \tag{4}$$

An analytic investigation of the function $f(t) = \nu t - t^\nu$ on the ray $(0, \infty)$ shows that for $\nu > 1$, the function f is strictly increasing on $(0, 1)$ and strictly decreasing on $(1, \infty)$. Also f attains its maximum $(= \nu - 1)$ at $t = 1$. Hence for $\nu > 1$ the inequality