Linear Algebra and its Applications 480 (2015) 127–143

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

L-rays of permutation matrices and doubly stochastic matrices

Geir Dahl

Department of Mathematics, University of Oslo, Norway

ARTICLE INFO

Article history:
Received 13 February 2015
Accepted 28 April 2015
Available online 16 May 2015
Submitted by R. Brualdi

MSC:
05B20
15A36

Keywords:
Permutation matrix
Doubly stochastic matrix
Majorization

ABSTRACT

Let M_n be the space of real $n \times n$ matrices. We investigate a linear transformation $\sigma : M_n \to \mathbb{R}^n$, called an *L-ray* (motivated by X-ray), which is defined in terms of sums of the entries in the blocks of a certain "L-shaped" partition of the positions of a matrix $A \in M_n$. We find descriptions of the image of the classes of permutation matrices and doubly stochastic matrices under this map, and show connections to majorization theory.

1. Introduction

Let M_n denote the vector space of real $n \times n$ matrices. A matrix in M_n is *doubly stochastic* (resp. *doubly substochastic*) if it is componentwise nonnegative and each row and column sum is 1 (resp. at most 1). A *subpermutation matrix* is a square $(0,1)$-matrix with at most one 1 in each row and column. For $1 \leq k \leq n$ define

E-mail address: geird@math.uio.no.

http://dx.doi.org/10.1016/j.laa.2015.04.033

$$L^{(k)} = \{(k,1),(k,2),\ldots,(k,k),(k-1,k),\ldots,(1,k)\}$$

which consists of the first k positions in row k and column k. Then $L^{(1)}, L^{(2)}, \ldots, L^{(n)}$ is a partition of $\{(i,j) : 1 \le i,j \le n\}$ and $\cup_{r=1}^{k} L^{(r)}$ is the set of positions of the leading $k \times k$ submatrix of an $n \times n$ matrix. For instance, for $n = 3$, $L^{(1)} = \{(1,1)\}$, $L^{(2)} = \{(2,1),(2,2),(1,2)\}$ and $L^{(3)} = \{(3,1),(3,2),(3,3),(2,3),(1,3)\}$.

Let A be a real $n \times n$ matrix. Define

$$\sigma_k(A) = \sum_{(i,j)\in L^{(k)}} a_{ij} \quad (1 \le k \le n)$$

and

$$\sigma(A) = (\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)). \tag{1}$$

Similarly, we define $\sigma_k^*(A) = \sum_{i,j\le k} a_{ij}$ $(1 \le k \le n)$, and $\sigma^*(A) = (\sigma_1^*(A), \sigma_2^*(A), \ldots, \sigma_n^*(A))$. Note that $\sigma_k^*(A)$ is the sum of all entries in the leading principal $k \times k$ submatrix A_k of A. Thus, if A is nonnegative, then $\sigma^*(A) = (\|A_1\|_s, \|A_2\|_s, \ldots, \|A_n\|_s)$ where $\|\cdot\|_s$ denotes the sum-norm; the sum of the absolute values of the entries in the matrix, see [11]. Since $\sigma_k^*(A) = \sum_{r=1}^{k} \sigma_r(A)$, there is a natural isomorphism which maps $\sigma(A)$ into $\sigma^*(A)$ (namely, $(x_1, x_2, \ldots, x_n) \to (x_1, x_1+x_2, \ldots, x_1+\cdots+x_n)$). This implies that our results concerning σ may be translated into similar results for σ^*.

We call $\sigma(A)$ the *L-ray* of A. This defines a linear transformation $\sigma : M_n \to \mathbb{R}^n$, given by $A \to \sigma(A)$. The term L-ray is inspired by the notion of X-ray in area of discrete tomography [9] (see below), and the sets $L^{(k)}$ are shaped like an L (backward). Several results were shown for Toeplitz and Hankel X-rays of permutation matrices in [6]; these X-rays are vectors of dimension $2n-1$ defined by summing entries along diagonals or anti-diagonals of a matrix of order n. Moreover, in [2] X-rays of anti-diagonals of permutation matrices were studied; different enumerative results and connections to other combinatorial objects were established.

A main goal in this paper is to investigate the image set

$$\sigma(\mathcal{A}) := \{\sigma(A) : A \in \mathcal{A}\} \tag{2}$$

for some matrix classes $\mathcal{A}$ in M_n. We call this the *L-ray problem*. In particular, we are interested in the following such classes $\mathcal{A}$: (i) $\mathbb{P}_n$; the set of permutation matrices of order n, (ii) Ω_n; the set of doubly stochastic matrices of order n, (iii) $\mathbb{P}_n^{sub}$ (resp. Ω_n^{sub}); the set of $n\times n$ subpermutation (resp. doubly substochastic) matrices, and (iv) subclasses of those above, but with constrained patterns.

Note that $\mathbb{P}_n \subseteq \Omega_n \subseteq \Omega_n^{sub}$, and therefore $\sigma(\mathbb{P}_n) \subseteq \sigma(\Omega_n) \subseteq \sigma(\Omega_n^{sub})$. In general, if $\mathcal{A}$ is a convex set (resp. a polyhedron), then $\sigma(\mathcal{A})$ is also convex (resp. a polyhedron). Therefore $\sigma(\Omega_n^{sub})$ and $\sigma(\Omega_n)$ are polyhedra.