Linear Algebra and its Applications 480 (2015) 144–167

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Spectral characterizations of signed lollipop graphs

Francesco Belardo [a,b,*], Paweł Petecki [a,c]

[a] University of Primorska – FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia
[b] Department of Mathematics and Computer Science, University of Messina, Italy
[c] AGH University of Science and Technology, Krakow, Poland

ARTICLE INFO

Article history:
Received 1 November 2014
Accepted 17 April 2015
Available online 16 May 2015
Submitted by R. Brualdi

MSC:
05C50
05C22

Keywords:
Signed lollipop graph
Adjacency matrix
Laplacian matrix
Spectral determination

ABSTRACT

Let $\Gamma = (G, \sigma)$ be a signed graph, where G is the underlying simple graph and $\sigma : E(G) \to \{+, -\}$ is the sign function on the edges of G. In this paper we consider the spectral characterization problem extended to the adjacency matrix and Laplacian matrix of signed graphs. After giving some basic results, we study the spectral determination of signed lollipop graphs, and we show that any signed lollipop graph is determined by the spectrum of its Laplacian matrix.

1. Introduction

Let $G = (V(G), E(G))$ be a graph of order $n = |V(G)| = |G|$ and size $m = |E(G)|$, and let $\sigma : E(G) \to \{+, -\}$ be a mapping defined on the edge set of G. Then $\Gamma = (G, \sigma)$ is a signed graph (sometimes called also sigraph). The graph G is its underlying graph, while σ its sign function (or signature). It is common to interpret the signs as the integers

* Corresponding author.
E-mail addresses: fbelardo@gmail.com (F. Belardo), pawel.petecki@gmail.com (P. Petecki).

http://dx.doi.org/10.1016/j.laa.2015.04.022
0024-3795/

$\{1, -1\}$. An edge e is positive (negative) if $\sigma(e) = 1$ (resp. $\sigma(e) = -1$). If $\sigma(e) = 1$ (resp. $\sigma(e) = -1$) for all edges in $E(G)$ then we write $(G, +)$ (resp. $(G, -)$). A cycle of Γ is said to be balanced, or positive, if it contains an even number of negative edges, otherwise the cycle is unbalanced, or negative. A signed graph is said to be balanced if all its cycles are balanced; otherwise, it is unbalanced. By $\sigma(\Gamma)$ we denote the product of signs of all cycles in Γ. Most of the concepts defined for graphs are directly extended to signed graphs. For example, the degree of a vertex v in G (denoted by $\deg(v)$) is also its degree in Γ. So $\Delta(G)$, the maximum (vertex) degree in G, also stands for $\Delta(\Gamma)$, interchangeably. Furthermore, if some subgraph of the underlying graph is observed, then the sign function for the subgraph is the restriction of the previous one. Thus, if $v \in V(G)$, then $\Gamma - v$ denotes the signed subgraph having $G - v$ as the underlying graph, while its signature is the restriction from $E(G)$ to $E(G - v)$ (all edges incident to v are deleted). Similar considerations hold for the disjoint union of signed graphs. If $U \subset V(G)$ then $\Gamma[U]$ denotes the signed induced subgraph of U, while $\Gamma - U = \Gamma[V(G) \setminus U]$. For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be (signature) switching equivalent to Γ. In fact, switching equivalent signed graphs can be considered as (switching) isomorphic graphs and their signatures are said to be equivalent. Switching equivalent graphs have the same set of positive cycles.

In the literature, simple graphs are studied by means of the eigenvalues of several matrices associated to graphs. The adjacency matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ whenever vertices i and j are adjacent and $a_{ij} = 0$ otherwise, is one of the most studied together with the Laplacian, or Kirchhoff, matrix $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}(\deg(v_1), \deg(v_2), \dots, \deg(v_n))$ is the diagonal matrix of vertex degrees. In the last 10 years another graph matrix has attracted the attention of many researchers, the so-called signless Laplacian matrix defined as $Q(G) = A(G) + D(G)$. Matrices can be associated to signed graphs, as well. The adjacency matrix $A(\Gamma) = (a^\sigma_{ij})$ with $a^\sigma_{ij} = \sigma(ij)a_{ij}$ is called the (signed) adjacency matrix and $L(\Gamma) = D(G) - A(\Gamma)$ is the corresponding Laplacian matrix. Both the adjacency and Laplacian matrices are real symmetric matrices, so the eigenvalues are real.

In this paper we shall consider both the characteristic polynomial of the adjacency matrix and of the Laplacian matrix of a signed graph Γ. Hence to avoid confusion we denote by

$$\phi(\Gamma, x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n,$$

the adjacency characteristic polynomial (or A-polynomial) whose roots, namely the adjacency eigenvalues (A-eigenvalues), are denoted by $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$. Similarly, for the Laplacian matrix, we denote by