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2-Local derivations on finite-dimensional Lie algebras

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ABSTRACT

We prove that every 2-local derivation on a finite-dimensional semi-simple Lie algebra $\mathcal{L}$ over an algebraically closed field of characteristic zero is a derivation. We also show that a finite-dimensional nilpotent Lie algebra $\mathcal{L}$ with $\dim \mathcal{L} \geq 2$ admits a 2-local derivation which is not a derivation.

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1. Introduction

The notion of 2-local derivations has been introduced by P. Šemrl in [7] and there the author described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on an infinite-dimensional separable Hilbert space H. Namely he has proved that every 2-local derivation on $B(H)$ is a derivation. A similar description for the finite-dimensional case appeared later in [6].

In [1] a new technique has been proposed which enabled the authors to generalize the above mentioned results of [6] and [7] for arbitrary Hilbert spaces H. It was proved that on the algebra $B(H)$ for an arbitrary Hilbert space H (no separability is assumed) every 2-local derivation is a derivation. A similar result for 2-local derivations on finite von Neumann algebras was obtained in [2]. In [3] the authors have extended all the above results for arbitrary semi-finite von Neumann algebras. Finally, in [4] it was proved that for any purely infinite von Neumann algebra M every 2-local derivation on M is a derivation. This completed the solution of the above problem for arbitrary von Neumann algebras.

In this paper we begin the study of 2-local derivations on non associative algebras, namely, we investigate 2-local derivations on finite-dimensional semi-simple Lie algebras over an algebraically closed field of characteristic zero.

All algebras and vector spaces considered in the paper are over an algebraically closed field $\mathbb{F}$ of characteristic zero.

2. 2-Local derivations on finite-dimensional semi-simple Lie algebras

The main result of this section is given as follows.

Theorem 2.1. *Let $\mathcal{L}$ be an arbitrary finite-dimensional semi-simple Lie algebra over $\mathbb{F}$. Then any* (*not necessarily linear*) *2-local derivation $T : \mathcal{L} \to \mathcal{L}$ is a derivation.*

Let $\mathcal{L}$ be a Lie algebra. The *center* of $\mathcal{L}$ is denoted by $Z(\mathcal{L})$:

$$Z(\mathcal{L}) = \{x \in \mathcal{L} : [x, y] = 0, \forall y \in \mathcal{L}\}.$$

A Lie algebra $\mathcal{L}$ is said to be *solvable* if $\mathcal{L}^{(k)} = \{0\}$ for some integer k, where $\mathcal{L}^{(0)} = \mathcal{L}$, $\mathcal{L}^{(k)} = [\mathcal{L}^{(k-1)}, \mathcal{L}^{(k-1)}]$, $k \geq 1$. Any Lie algebra $\mathcal{L}$ contains a unique maximal solvable ideal, called the radical of $\mathcal{L}$ and denoted by $\operatorname{Rad}\mathcal{L}$. A non trivial Lie algebra $\mathcal{L}$ is called *semi-simple* if $\operatorname{Rad}\mathcal{L} = 0$. That is equivalent to requiring that $\mathcal{L}$ has no nonzero abelian ideals.

Given a vector space V, let $\mathfrak{gl}(V)$ denote the Lie algebra of all linear endomorphisms of V. A representation of a Lie algebra $\mathcal{L}$ on V is a Lie algebra homomorphism $\rho : \mathcal{L} \to \mathfrak{gl}(V)$. For example, $\operatorname{ad} : \mathcal{L} \to \mathfrak{gl}(\mathcal{L})$ given by $\operatorname{ad}(x)(y) = [x, y]$ is a representation of $\mathcal{L}$ on the vector space $\mathcal{L}$ called the adjoint representation. If V is a finite-dimensional vector space then the representation ρ is said to be finite-dimensional.