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Nonscalar matrix polynomial representation of some scalar polynomials

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ABSTRACT

We show how a class of scalar polynomials can be expressed as nonscalar matrix polynomials to which matrix methods can be applied that yield improvements over scalar results and generate new ones. To demonstrate the advantages of this approach, we use it to compute bounds for the zeros of those polynomials.

1. Introduction

Polynomials are usually treated as the scalar objects that they are, but here we will view them as objects of a higher dimension, namely, as matrix polynomials, i.e., polynomials whose coefficients are matrices. These occur in polynomial eigenvalue problems,

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where a nonzero vector v and a $\lambda \in \mathbb{C} \cup \{\infty\}$ are sought such that $P(\lambda)v = 0$, with

$$P(\lambda) = A_n\lambda^n + A_{n-1}\lambda^{n-1} + \cdots + A_0 ,$$

where the coefficients A_i are all $m \times m$ complex matrices and $detP$ is not identically zero (i.e., P is regular). If A_n is singular then there are infinite eigenvalues and if A_0 is singular then zero is an eigenvalue. There are nm eigenvalues, including possibly infinite ones. The finite eigenvalues are the solutions of $detP(\lambda) = 0$. Such problems are encountered in several engineering applications (e.g., [7,19] and references therein).

Our goal is to demonstrate that the idea of representing a scalar polynomial as the determinant of a matrix polynomial of size ≥ 2 is worthwile and can lead to improvements over standard (scalar) results. Our focus is on a methodology and for practical purposes we will consider a restricted class of polynomials. However, this class is sufficient to motivate the idea while its simplicity keeps the results transparent.

There are infinitely many ways to write a polynomial as the determinant of a matrix polynomial and the advantages of one over the other are not a priori clear. One systematic way to achieve this that we have found to be both convenient and amenable to generalization is illustrated by the following example of a simple lacunary polynomial $p(z) = z^n + a_2z^2 + a_1z + a_0$ with $n \geq 4$ an even integer and $a_0a_2 \neq 0$. Denoting the (nonzero) roots of its quadratic tail by ρ_1 and ρ_2, p can be written as $p(z) = z^n + a_2(z - \rho_1)(z - \rho_2)$, so that

$$p(z) = \det \begin{pmatrix} z^{n/2} & i\sqrt{a_2}z - i\sqrt{a_2}\rho_1 \\ i\sqrt{a_2}z - i\sqrt{a_2}\rho_2 & z^{n/2} \end{pmatrix} .$$

This determinant can then be expressed as

$$\det \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^{n/2} + i\sqrt{a_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z - i\sqrt{a_2} \begin{pmatrix} 0 & \rho_1 \\ \rho_2 & 0 \end{pmatrix} \right] \equiv \det P(z) . \tag{1}$$

The order of the matrix polynomial P is half that of p, its coefficients are 2×2 matrices and both its leading and trailing coefficients are nonsingular. It therefore has n nonzero finite eigenvalues that coincide with the zeros of p. One can now apply to P the matrix versions of any scalar results that would normally be applied to p.

An important current topic of interest in matrix polynomials is the concept of ℓ-ification [5], a generalization of linearization, which becomes especially important when mere linearization cannot preserve the structure present in the matrix polynomial. Any such ℓ-ification of a scalar polynomial also leads to a matrix polynomial in the sense that was explained above, but the converse is not true. We refer to [5] for a thorough and self-contained treatment of this topic.

We recall that the zeros of the complex monic scalar polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and the eigenvalues of the monic matrix polynomial $P(z) = Iz^n + A_{n-1}z^{n-1} + \cdots + A_1z + A_0$, with $A_j \in \mathbb{C}^{m\times m}$ for $j = 0, \ldots, n-1$ and I the