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Some results on the structure and spectra of matrix-products

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ABSTRACT

We consider certain matrix-products where successive matrices in the product belong alternately to a particular qualitative class or its transpose. The main theorems relate structural and spectral properties of these matrix-products to the structure of underlying bipartite graphs. One consequence is a characterisation of caterpillars: a graph is a caterpillar if and only if all matrix-products associated with it have real nonnegative spectrum. Several other equivalences of this kind are proved. The work is inspired by certain questions in dynamical systems where such products arise naturally as Jacobian matrices, and the results have implications for the existence and stability of equilibria in these systems.

1. Introduction and statement of the main results

The question of how the structure of a matrix in a combinatorial sense relates to its linear algebraic properties has been intensively studied, particularly in the context of

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sign nonsingularity ([1–4] to name just a few examples), but also of other questions in linear algebra, both spectral and nonspectral ([5–7] for example). Here we explore how the combinatorial structure of a real matrix A, not necessarily square, as encoded in its bipartite graph, relates to properties of matrix-products where successive matrices in the product belong alternately to the qualitative class of A or its transpose.

The results are inspired partly by the study of chemical reaction networks, namely dynamical systems describing the evolution of chemical species undergoing a set of reactions. In this setting the matrices studied are Jacobian matrices, and the bipartite graph from which one wishes to draw conclusions is a natural representation of the chemical system, often termed the "species-reaction graph" or "SR-graph" [8,9]. Under weak assumptions, systems of chemical reactions have Jacobian matrices which factorise as $-AB^{\mathrm{t}}$ where A and B are matrices such that B lies in the closure of the qualitative class of A. The study of these systems thus naturally raises general questions about what can be said about matrix-products where alternate factors belong either to some qualitative class or its transpose.

The main results, Theorems 1 to 3, are easy to state after some definitions.

Definition 1.1 *(Sign-pattern, qualitative class).* Given $A \in \mathbb{R}^{n\times m}$ define $\mathrm{sign}\, A \in \mathbb{R}^{n\times m}$, the sign-pattern of A, as the $(0, 1, -1)$-matrix whose entries have the same signs as the entries of A; the qualitative class of A is the set of matrices with the same sign-pattern as A, i.e., $\mathcal{Q}(A) = \{B \in \mathbb{R}^{n\times m} : \mathrm{sign}\, B = \mathrm{sign}\, A\}$. Also useful is $\mathcal{Q}_0(A)$, the topological closure of $\mathcal{Q}(A)$, regarded as a subset of $\mathbb{R}^{n\times m}$.

Definition 1.2. Given $A \in \mathbb{R}^{n\times m}$, define

$$\mathcal{Q}^k(A) = \{A_1A_2^{\mathrm{t}}A_3\cdots A_k^{\mathrm{t}} : A_i \in \mathcal{Q}(A),\ i = 1,\ldots,k\},$$
$$\mathcal{Q}_0^k(A) = \{A_1A_2^{\mathrm{t}}A_3\cdots A_k^{\mathrm{t}} : A_i \in \mathcal{Q}_0(A),\ i = 1,\ldots,k\}.$$

Note that $\mathcal{Q}^k(A) \subseteq \mathcal{Q}_0^k(A) \subseteq \mathrm{cl}(\mathcal{Q}^k(A))$ where $\mathrm{cl}(\mathcal{Q}^k(A))$ is the closure of $\mathcal{Q}^k(A)$.

Definition 1.3 *(P_0-matrices).* A real square matrix is a P_0-matrix (resp. P-matrix) if all of its principal minors are nonnegative (resp. positive). We write $\mathbf{P_0}$ for the set of P_0-matrices.

Definition 1.4 *(Matrices with nonnegative real eigenvalues).* We write **PS** for the set of real matrices with real nonnegative spectrum. (Real) positive semidefinite matrices are the symmetric elements of **PS**.

Definition 1.5 *(Forest, caterpillar forest).* A forest is an acyclic graph; a **caterpillar** is a tree which becomes a path on removal of its leaves; a caterpillar forest is a forest each of whose connected components is a caterpillar.