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Bounds on polygons of higher rank numerical ranges ☆

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ABSTRACT

Research in higher rank numerical ranges has originally been motivated by problems in quantum information theory, particularly in quantum error correction. The higher rank numerical range generalizes the classical numerical range of an operator. The higher rank numerical range is typically not a polygon, however when we consider normal operators the higher rank numerical range is a polygon in the complex plane $\mathbb{C}$. In this article, we give a new proof of an upper bound on the number of sides of the higher-rank numerical range of a normal operator and we also find a lower bound for the number of sides of the higher-rank numerical range of a unitary operator. We show that these bounds are the best possible.

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1. Introduction

In this paper we will use $\mathcal{H}$ to refer to a finite dimensional Hilbert space and $\mathcal{L}(\mathcal{H})$ to refer to the space of linear operators on $\mathcal{H}$.

We begin by recalling the definition of the classical numerical range.

Definition 1.1 *(Classical numerical range).* (See [7].) The (classical) numerical range of an operator $A \in \mathcal{L}(\mathcal{H})$ is the set of complex scalars that satisfy

$$W(A) = \{\langle Ax, x\rangle \mid x \in \mathcal{H},\ \|x\| = 1\}. \tag{1}$$

For an operator $A \in \mathcal{L}(\mathcal{H})$ the following properties of $W(A)$ are immediate [7]:

1. $W(\alpha I + \beta A) = \alpha + \beta W(A)$ for all complex scalars α, β and identity I.
2. $W(A^*) = \{\bar{\mu} \mid \mu \in W(A)\}$.
3. $W(U^*AU) = W(A)$ for all unitary U.

Geometrically the classical numerical range of an operator A can be considered as the intersection of closed half-planes given by

$$\{\lambda \in \mathbb{C} \mid e^{i\theta}\lambda + e^{-i\theta}\bar{\lambda} \leq \lambda_1(e^{i\theta}A + e^{-i\theta}A^*)\} \tag{2}$$

where $\theta \in [0, 2\pi)$ and $\lambda_1(H)$ is the largest eigenvalue of the Hermitian matrix H [9]. This geometric interpretation supplies one way to prove the following classical result:

Theorem 1.2 *(Toeplitz–Hausdorff). The numerical range of an operator is convex.*

We are now ready to introduce the higher rank numerical range.

Definition 1.3 *(The rank-k numerical range of an operator).* (See [3].) Let k be a positive integer, let $\mathcal{H}$ be at least of dimension k and let $\mathcal{P}_k$ denote the set of all rank-k projections on $\mathcal{H}$. For $A \in \mathcal{L}(\mathcal{H})$ the rank-k numerical range of A is defined as

$$\Lambda_k(A) := \{\lambda \in \mathbb{C} \mid PAP = \lambda P \textit{ for some } P \in \mathcal{P}_k\}. \tag{3}$$

Remark 1.4. The complex scalars λ in Definition 1.3 are referred to as “compression values” in the literature [3].

Remark 1.5. When $k = 1$ we get $\Lambda_1(A) = W(A)$ [4].

Lemma 1.6. *(See [4].) The following set inclusion holds for higher rank numerical ranges:*

$$\Lambda_1(A) \supseteq \Lambda_2(A) \supseteq \Lambda_3(A) \supseteq \ldots \supseteq \Lambda_N(A) \tag{4}$$