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A complete solution of the permutability problem for Toeplitz and Hankel matrices

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ABSTRACT

Conditions under which two Toeplitz matrices commute are known since at least 1998. For a pair of Hankel matrices, the corresponding commutativity conditions were recently obtained by Gel'fgat. In this paper, we complete our analysis of the much more complex problem of characterizing matrix pairs (T, H) such that T is a Toeplitz matrix, H is a Hankel matrix, and $TH = HT$. The only case left open in our previous publications on this subject is the one where the Toeplitz matrix T is centrosymmetric. Here, we present an exhaustive study of this case, which yields a complete solution of the above commutativity problem.

1. Introduction

Let $M_n(\mathbf{C})$ be the set of complex $n \times n$ matrices. In this paper, we deal with Toeplitz matrices, which are usually written in the form

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$$T = \begin{pmatrix} t_0 & t_1 & t_2 & \dots & t_{n-1} \\ t_{-1} & t_0 & t_1 & \dots & t_{n-2} \\ t_{-2} & t_{-1} & t_0 & \dots & t_{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ t_{-n+1} & t_{-n+2} & t_{-n+3} & \dots & t_0 \end{pmatrix}, \tag{1}$$

and Hankel matrices written as

$$H = \begin{pmatrix} h_{n-1} & h_{n-2} & h_{n-3} & \dots & h_0 \\ h_{n-2} & h_{n-3} & h_{n-4} & \dots & h_{-1} \\ h_{n-3} & h_{n-4} & h_{n-5} & \dots & h_{-2} \\ \dots & \dots & \dots & \dots & \dots \\ h_0 & h_{-1} & h_{-2} & \dots & h_{-n+1} \end{pmatrix}. \tag{2}$$

We will often use certain special Toeplitz and Hankel matrices. Recall that the Toeplitz matrix (1) is a circulant if

$$t_{-j} = t_{n-j}, \qquad j = 1, 2, \dots, n-1,$$

a skew-circulant if

$$t_{-j} = -t_{n-j}, \qquad j = 1, 2, \dots, n-1,$$

and a ϕ-circulant if

$$t_{-j} = \phi t_{n-j}, \qquad j = 1, 2, \dots, n-1.$$

Let $\mathcal{P}_n$ be the backward identity matrix of order n:

$$\mathcal{P}_n = \begin{pmatrix} & & 1 \\ & \dots & \\ 1 & & \end{pmatrix}.$$

If T is a Toeplitz matrix, then, obviously,

$$H = T\mathcal{P}_n \tag{3}$$

is a Hankel matrix, and vice versa. We call matrix (3) a Hankel circulant, skew-circulant, or ϕ-circulant if T is a (conventional) circulant, skew-circulant, or ϕ-circulant, respectively.

Conditions under which Toeplitz matrices T_1 and T_2 are permutable are known at least since 1998 (e.g., see [1]). These conditions are closely related to the problem of characterizing normal Toeplitz matrices, which was solved by these authors at about the same time (see [2]).