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Fractional differential operators in the complex matrix-variate case

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ABSTRACT

There are many applications for fractional integrals, fractional derivatives and fractional differential equations in the real scalar variable cases. However, when it comes to fractional integrals in the real and complex matrix-variate cases there are not many papers. The author has given some definitions and properties for fractional integrals and fractional derivatives in the real matrix-variate cases recently. Some situations of fractional derivatives in the complex matrix-variate case are discussed in the present article, along with the definitions for fractional derivatives in the complex matrix-variate case. The definition introduced here enables us to handle certain types of derivatives of real-valued scalar functions of matrix argument in the complex domain. It is not universally applicable. Fractional derivatives in the Riemann–Liouville and Caputo senses are evaluated when the arbitrary function is compatible with right and left sided fractional integrals in the complex matrix-variate cases. Kober operators of the first and second kind in the complex matrix-variate cases are also examined here.

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1. Introduction

Several papers deal with fractional integrals and fractional derivatives in the real scalar variable case. For their applications in stochastic processes and random walk problems, see for example [1,2]. Several papers deal with solutions of fractional differential equations in the real variable case, see for example [3,14]. Fractional integrals in the matrix-variate case are given by the author recently. Some discussions on functions of matrix argument may be seen from [4–13]. Fractional integrals in the real matrix-variate case may be seen from [8,12] and those in the complex case may be seen from [7,8]. Some results on fractional derivatives in the real matrix-variate case may be seen from [4,9]. Here we introduce fractional differential operators in the complex matrix-variate case, which are applicable when the arbitrary function of matrix argument has certain structures. Kober operators of the first and second kinds in the complex domain are also discussed in the Riemann–Liouville and Caputo senses.

The following standard notations will be used. All matrices appearing here are $p \times p$ hermitian positive definite when in the complex domain and positive definite in the real case, unless otherwise stated. $\mathrm{tr}(\cdot)$ and $\det(\cdot)$ denote the trace and determinant of the square matrix $(\cdot)$ respectively. $|\det(\cdot)|$ denotes the absolute value of the determinant of $(\cdot)$. For example, if $\det(B) = c + id$, $i = \sqrt{-1}$ and c and d are real scalars then

$$|\det(B)| = [(c + id)(c - id)]^{\frac{1}{2}} = [c^2 + d^2]^{\frac{1}{2}} = [\det(BB^*)]^{\frac{1}{2}} \tag{1.1}$$

where B^* is the conjugate transpose of B. If $X = (x_{ij})$ is $p \times q$ and real then $\mathrm{d}X$ will denote the wedge product of all differentials

$$\mathrm{d}X = \prod_{i=1}^{p}\prod_{j=1}^{q} \wedge \mathrm{d}x_{ij}, \quad \mathrm{d}X = \prod_{i \geq j=1}^{p} \wedge \mathrm{d}x_{ij} \text{ for } X = X',\ p \times p \text{ and real.} \tag{1.2}$$

If $\tilde{X} = X_1 + iX_2$, $i = \sqrt{-1}$ where X_1 and X_2 are real matrices then $\mathrm{d}\tilde{X} = \mathrm{d}X_1 \wedge \mathrm{d}X_2$. Matrix variables in the complex domain will be denoted by a tilde as $\tilde{X}$. Real matrix variables X and real or complex constant matrices will be written without a tilde. The following standard notations for real and complex matrix-variate gamma function will be used. Matrix-variate gamma in the real case will be denoted by $\Gamma_p(\alpha)$ where $\Gamma_p(\alpha)$ has the following expression and integral representation:

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma(\alpha - \frac{1}{2})\ldots\Gamma(\alpha - \frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2} \tag{1.3}$$

$$\Gamma_p(\alpha) = \int_{X>O} [\det(X)]^{\alpha - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X, \tag{1.4}$$

where $X > O$ means X is positive definite and $\Re(\alpha)$ denotes the real part of α. The integration is done over all real positive definite matrices X. In the complex domain,