Linear Algebra Appl. 478 (2015) 218–227

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

B-majorization and its linear preservers

Mohammad Ali Hadian Nadoshan, Ali Armandnejad *

Department of Mathematics, Vali-e-Asr University of Rafsanjan, P.O. Box: 7713936417, Rafsanjan, Iran

ARTICLE INFO

Article history:
Received 27 January 2015
Accepted 26 March 2015
Available online 15 April 2015
Submitted by R. Brualdi

MSC:
15A04
15A21

Keywords:
B-majorization
Doubly stochastic matrix
Linear preserver

ABSTRACT

In this paper the concept of B-majorization is studied and some facts about this relation are found. Next, the linear preservers of B-majorization on $\mathbb{R}^n$ and $M_{n,m}$ will be characterized. It is also shown that the nonzero linear preservers and strong linear preservers of B-majorization on $\mathbb{R}^n$ are the same.

1. Introduction

Let $x = (x_1, \ldots, x_n)^t$ and $y = (y_1, \ldots, y_n)^t$ be two vectors in $\mathbb{R}^n$ and let $x^\downarrow = \left(x_1^\downarrow, \ldots, x_n^\downarrow\right)$ be the vector obtained by rearranging the coordinates of x in decreasing order, i.e. $x_1^\downarrow \geq x_2^\downarrow \geq \ldots \geq x_n^\downarrow$. Then x is said to be majorized by y (written $x \prec y$) if

$$\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow \qquad k = 1, 2, \ldots, n \tag{1.1}$$

* Corresponding author.
E-mail addresses: ma.hadiann@stu.vru.ac.ir (M.A. Hadian Nadoshan), armandnejad@vru.ac.ir (A. Armandnejad).

http://dx.doi.org/10.1016/j.laa.2015.03.036

and $\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}$. Respectively x is said to be weakly majorized by y (written $x \prec_w y$) if x and y satisfy (1.1), see [10].

A nonnegative matrix is doubly stochastic if the sum of all entries in each row and each column is 1. Respectively a nonnegative matrix is doubly substochastic if the sum of all entries in each row and each column is less than or equal to 1. It is a well known fact that every doubly stochastic matrix is a convex combination of finitely many permutation matrices.

Theorem 1.1 *(Birkhoff's theorem). The set of $n \times n$ doubly stochastic matrices is a convex set whose extreme points are the permutation matrices.*

Respectively a doubly substochastic matrix is a convex combination of finitely many matrices having at most one entry 1 in each row and each column and all other entries zero.

We know that $x \prec y$ if and only if $x = Dy$ for some doubly stochastic matrix D. Respectively for nonnegative x and y, $x \prec_w y$ if and only if $x = Dy$ for some doubly substochastic matrix D, see [2] and [5].

By $|A|$ we mean $[|a_{ij}|]$ for each $A = [a_{ij}] \in M_{n,m}$. If $x \prec y$ and $y \prec x$ we write $x \sim y$.

2. B-majorization

Let V be a finite dimensional real vector space, $\mathrm{O}(V)$ the orthogonal group acting on V and G a closed subgroup of $\mathrm{O}(V)$. The group G induces an equivalence relation on V, defined by $x \approx y$ if and only if $y = gx$ for some $g \in G$. The equivalence classes of this relation are called the orbits of G. For each $y \in V$ the orbit of y is as follows:

$$\mathrm{O}_G(y) = \{gy | g \in G\}.$$

A vector x is said to be G-majorized by y if $x \in conv(\mathrm{O}_G(y))$, where the notation $conv(A)$ is the convex hull of a set A.

Definition 2.1. A signed permutation matrix is a matrix with exactly one entry $+1$ or -1 in each row and each column and all other entries zero.

We denote the set of all signed permutation matrices by S'_n. If $G = S_n$ is the group of permutation matrices, we reach to the classical majorization. The Coxeter group of type B_n is the group of signed permutations of n letters. It can be represented by $n \times n$ signed permutation matrices. For more information see [4,7]. The majorization induced by group B_n is called the B-majorization and is denoted by $\prec_{\mathrm{B}}$.

Definition 2.2. A real matrix $D = [d_{ij}] \in M_{n,m}$ is called generalized substochastic if $\sum_{j=1}^{m} |d_{ij}| \leq 1$ for every i $(1 \leq i \leq n)$ and $\sum_{i=1}^{n} |d_{ij}| \leq 1$ for every j $(1 \leq j \leq m)$.