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Inverse of the distance matrix of a cactoid digraph

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ABSTRACT

A strongly connected directed graph in which any two directed cycles of it share at most one common vertex is called a cactoid digraph. Let D be its distance matrix. By a theorem of Graham, Hoffman and Hosoya, we have $\det(D) \neq 0$. In this paper, we give the formulas for both the determinant and the inverse of the distance matrix D for a cactoid digraph.

1. Introduction

Graham et al. [6] proved a very attractive theorem about the determinant of the distance matrix $D(G)$ of a strongly connected digraph G as a function of the distance matrices of its blocks. In a strongly connected digraph, the *distance* $d(u, v)$ from vertex u to vertex v is the length of a shortest directed path from vertex u to v. The *distance matrix* $D(G)$ of a strongly connected digraph G is a matrix with (u, v)-entry $d(u, v)$. Let

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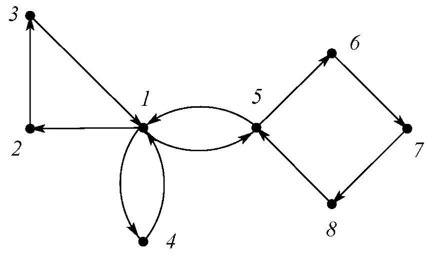

Fig. 1. An example of cactoid digraph with four directed cycles.

A be an $n \times n$ matrix. Recall that the cofactor c_{ij} of corresponding to (i,j)-entry of A is defined as $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting i-row and j-column of A, and $Cof(A) = \sum_{i,j} c_{i,j}$ is the sum of all cofactors of A. Graham et al. [6] showed the following interesting theorem.

Theorem 1.1. *If G is a strongly connected digraph with blocks $G_1, G_2, ..., G_r$, then*

$$Cof(D(G)) = \prod_{i=1}^{r} Cof(D(G_i)),$$

$$\det(D(G)) = \sum_{i=1}^{r} \det(D(G_i)) \prod_{j \neq i} Cof(D(G_j)).$$

In this paper, all digraphs are simple (no loops and no parallel-arcs). A strongly connected digraph G is called *cactoid digraph* if any two directed cycles of G share at most one common vertex and G_i $(1 \leq i \leq r)$ denotes all directed cycles of G. See Fig. 1 for an example. Further, by Theorem 1.1, we can give a formula (Theorem 2.3 in Section 2) for the determinant $\det(D(G))$ for the distance matrix $D(G)$ of a cactoid digraph G in terms of the sizes of its directed cycles. From the formula it will be clear that $\det(D(G)) \neq 0$. Hence, we are interested in finding the inverse matrix $D(G)^{-1}$.

For the case when lengths of all directed cycles of G are 2 (a cactoid digraph can be viewed as an undirected graph and it is a tree in fact), it is known [1,7] that $D(G)^{-1} = -\frac{L(G)}{2} + \frac{1}{2(n-1)}\tau\tau^T$, where $L(G)$ is the Laplacian matrix of G and τ is the n-dimensional column vector with $\tau(v) = 2 - deg(v)$. For the case of undirected graphs, there are many results about the inverse and determinant of the distance matrix of a graph whose blocks are cliques. See [2,4,5,8] for undirected graphs and [3] for bidirected tree.

2. The inverse and determinant of *D*

We begin by investigating the distance matrix for a directed cycle.

Let C_p be a directed cycle on p vertices. Then the distance matrix $D(C_p)$ of C_p is a circulant matrix as follows: