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Perturbation analysis of the extinction probability of a Markovian binary tree ☆

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A B S T R A C T

The extinction probability of the Markovian Binary Tree (MBT) is the minimal nonnegative solution of a Quadratic Vector Equation (QVE). In this paper, we present a perturbation analysis for the extinction probability of a supercritical MBT. We derive a perturbation bound for the minimal nonnegative solution of the QVE, which is a bound on the difference between the solutions of two nearby equations in terms of the perturbation magnitude. A posteriori error bound is also given, which is a bound on the distance between an approximate solution and the real solution, in terms of the residual of the approximate solution. Numerical experiments show that these bounds are fairly sharp.

1. Introduction

We first introduce necessary notation for this paper. For matrices $A = [a_{ij}], B = [b_{ij}] \in \mathbb{R}^{m\times n}$, we write $A \geq B$ ($A > B$) if $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$) holds for all i, j. For

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vectors $x, y \in \mathbb{R}^n$, we write $x \geq y$ ($x > y$) if $x_i \geq y_i$ ($x_i > y_i$) holds for all $i = 1, 2, \ldots, n$. The spectral radius of a square matrix A is denoted by $\rho(A)$. The symbol $\|\cdot\|$ will be used to denote the infinity-norm unless we have a special statement. The column vector of all ones is denoted by e, i.e., $e = (1, 1, \ldots, 1)^\top$. The symbol $\otimes$ denotes the Kronecker product.

In this paper, we consider the perturbation analysis of the extinction probability of a Markovian Binary Tree (MBT). MBTs belong to a special class of continuous-time Markovian multi-type branching processes [1], which are used to model the growth of populations consisting of several types of individuals who may reproduce and die during their lifetime. Applications have been found in biology and epidemiology [4,9], and telecommunication systems [8,15]. We refer the readers to [2,5,7,10] for a detailed description of MBTs.

In MBTs, the individuals give birth to only one child at a time and the life of each individual is controlled by a Markovian process, called the phase process, on the space of transient states $\{1, 2, \ldots, n\}$. An important issue is the computation of the extinction probability of the population. It is shown in [2] that the extinction probability is the minimal nonnegative solution (in the componentwise ordering) of the following Quadratic Vector Equation (QVE):

$$x = a + B(x \otimes x), \tag{1.1}$$

where $x = [x_i] \in \mathbb{R}^n$ is the unknown vector with x_i the extinction probability of a population starting from an individual in state i, $a = [a_i] \in \mathbb{R}^n$ is the coefficient vector of QVE (1.1) with a_i the probability that an individual in state i dies out without producing offspring, $B = [B_{i,n(j-1)+k}] \in \mathbb{R}^{n \times n^2}$ is the coefficient matrix of QVE (1.1) with $B_{i,n(j-1)+k}$ the probability that an individual in phase i eventually produces a child in state j and the parent switches to phase k after the birth. Using the fact that the probabilities of all possible outcomes for an individual in state i must sum to 1, for $i = 1, 2, \cdots, n$, we know that the vector e must be a solution of (1.1), i.e.,

$$e = a + B(e \otimes e). \tag{1.2}$$

Let

$$M = B(I \otimes e + e \otimes I). \tag{1.3}$$

An MBT is irreducible if the mean progeny matrix M is irreducible, that is if for each $1 \leq i, j \leq n$, there exists an integer $k \geq 0$ such that $(M^k)_{ij} > 0$. An MBT is called subcritical, supercritical, or critical if $\rho(M)$ is strictly less than 1, strictly greater than 1, or equal to 1, respectively [1]. In the subcritical and critical cases, the minimal nonnegative solution of (1.1) is the vector of all ones e, while in the supercritical case, the minimal nonnegative solution x^* satisfies $x^* \leq e, x^* \neq e$ [7]. Hereafter, we will only concentrate on the supercritical irreducible MBT.