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Three short descriptions of the symmetric and of the skew-symmetric solution set

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A R T I C L E I N F O

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Similarly to various descriptions of the solution set S of linear systems of equations with perturbed input data we present several short descriptions for the symmetric solution set S_{sym} and the skew-symmetric solution set S_{skew}.

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1. Introduction

In this short note we consider the symmetric solution set

$$S_{\rm sym} = \{\, x \in \mathbb{R}^n \mid Ax = b,\ A = A^T \in [A] = [A]^T,\ b \in [b]\,\},$$

and the skew-symmetric solution set

$$S_{\rm skew} = \{\, x \in \mathbb{R}^n \mid Ax = b,\ A = -A^T \in [A] = -[A]^T,\ b \in [b]\,\},$$

where $[A] = ([a]_{ij})$ denotes a given $n \times n$ interval matrix and $[b] = ([b]_i)$ denotes a given interval vector with n components. In the skew-symmetric case we assume, in addition, $[a]_{ii} = 0,\ i = 1, \dots, n$. Obviously, both solution sets are subsets of the general solution set $S = \{\, x \in \mathbb{R}^n \mid Ax = b,\ A \in [A],\ b \in [b]\,\}$, where the restrictions on A and $[A]$ are dropped. All three solution sets can be interpreted as the set of all solutions of linear systems with perturbed input data; cf. for instance [6] or [8], where also references to the origin of such systems are given. The general solution set S can be characterized in various ways three of which are presented by Oettli and Prager in [7], by Beeck in [2], and by Hartfiel in [3]. In our paper we will show that $S_{\rm sym}$ and $S_{\rm skew}$ can be characterized similarly. To this end we need some basic knowledge in interval analysis as can be found in the usual textbooks on this subject. We write nonempty real compact intervals $[a] = [\underline{a}, \bar{a}]$ within square brackets with the exception of point intervals, and we denote the set of these intervals by $\mathbb{IR}$. With interval vectors and interval matrices we proceed similarly. Midpoint, radius, and absolute value of an interval $[a]$ are denoted by $\check{a} = {\rm mid}([a]) = (\underline{a} + \bar{a})/2$, ${\rm rad}([a]) = (\bar{a} - \underline{a})/2$, and $|[a]| = \max\{|\underline{a}|, |\bar{a}|\}$, respectively. For vectors and matrices, these operations are applied entrywise. For $p \in \{0,1\}^n$ we define the complementary vector $\bar{p} = (1, 1, \dots, 1)^T - p$ and the diagonal matrix $D_p = {\rm diag}(p)$ with $d_{ii} = p_i,\ i = 1, \dots, n$. As a particular vector p we use the i-th column $e^{(i)}$ of the $n \times n$ identity matrix I. We apply the symbol $\prec_{\rm lex}$ to denote strict lexicographic ordering of vectors, i.e., $p \prec_{\rm lex} q$ if for some k we have $p_i = q_i$, $i < k$, and $p_k < q_k$.

2. Results

First we present our result on $S_{\rm sym}$ emphasizing that formula (2) with the restriction (1) is a repetition of a result in [6].

Theorem 1. *Let* $[A] = [A]^T \in \mathbb{IR}^{n \times n},\ [b] \in \mathbb{IR}^n,\ x \in \mathbb{R}^n,\ r = \check{b} - \check{A}x$. *Then* $x \in S_{\rm sym}$ *if and only if* $x \in S$ *and if for all vectors* $p, q \in \{0,1\}^n$ *with*

$$0 \ne p \prec_{\rm lex} q \quad and \quad p^T q = 0 \tag{1}$$

one of the following three relations (2), (3), (4) is satisfied: