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Zero Jordan product determined algebras

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A B S T R A C T

We prove that a unital algebra $\mathcal{A}$ over a field of characteristic not 2 is zero Jordan product determined if it is generated by idempotents. Since an example of such an algebra is the matrix algebra $M_n(\mathcal{B})$ where $n \geqslant 2$ and $\mathcal{B}$ is any unital algebra, this yields answers to questions posed in [4, p. 1492] and [7, p. 117].

1. Introduction

Throughout this paper, let $\mathcal{A}$ be a unital associative algebra over $\mathbb{F}$ and $\mathcal{X}$ be a linear space over $\mathbb{F}$, where $\mathbb{F}$ is a field of characteristic not 2. In an algebra $\mathcal{A}$, we can define the Jordan product by $a \circ b = ab + ba$ for each a, b in $\mathcal{A}$.

$\mathcal{A}$ is said to be *zero product determined* if every bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into any linear space $\mathcal{X}$ satisfying

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$$\phi(a,b)=0,\ \text{whenever } ab=0$$

can be written as $\phi(a,b)=T(ab)$, for some linear mapping T from $\mathcal{A}$ into $\mathcal{X}$.

Similarly $\mathcal{A}$ is said to be *zero Jordan product determined* if every bilinear mapping ϕ from $\mathcal{A}\times\mathcal{A}$ into any linear space $\mathcal{X}$ satisfying

$$\phi(a,b)=0,\ \text{whenever } a\circ b=0$$

can be written as $\phi(a,b)=T(a\circ b)$, for some linear mapping T from $\mathcal{A}$ into $\mathcal{X}$.

We denote by $\mathfrak{L}(\mathcal{A})$ the linear span of all idempotents in $\mathcal{A}$, and by $\mathfrak{J}(\mathcal{A})$ the subalgebra of $\mathcal{A}$ generated by all idempotents in $\mathcal{A}$.

In [1,3–7], several authors study bilinear mappings through their action on zero product or zero Jordan product.

In [3], Brešar shows that if $\mathcal{A}=\mathfrak{J}(\mathcal{A})$, then $\mathcal{A}$ is zero product determined. In [5], Ghahramani proves that if $\mathcal{A}=\mathfrak{L}(\mathcal{A})$, then $\mathcal{A}$ is zero Jordan product determined.

In [4], Brešar et al. show that the matrix algebra $M_n(\mathcal{B})$ of $n\times n$ matrices over a unital algebra $\mathcal{B}$ with $\frac{1}{2}$ is zero Jordan product determined for every $n\geqslant 3$. They ask whether the result is true for $n=2$.

In Section 2, we improve the results in [3,5] through studying bilinear mappings on an algebra $\mathcal{A}$ and show that $\mathcal{A}$ is zero Jordan product determined if $\mathcal{A}=\mathfrak{J}(\mathcal{A})$. As applications, we affirmatively answer two questions posed in [4, p. 1492] and [7, p. 117].

2. Main results

Theorem 2.1. *If ϕ is a bilinear mapping from $\mathcal{A}\times\mathcal{A}$ into $\mathcal{X}$ such that*

$$a\circ b=0\Rightarrow\phi(a,b)=0$$

for all a, b in $\mathcal{A}$, then

$$\phi(a,x)=\frac{1}{2}\phi(ax,1)+\frac{1}{2}\phi(xa,1)$$

for all a in $\mathcal{A}$ and x in $\mathfrak{J}(\mathcal{A})$. Thus $\mathcal{A}$ is zero Jordan product determined if $\mathcal{A}=\mathfrak{J}(\mathcal{A})$.

Proof. By the definition of $\mathfrak{J}(\mathcal{A})$, we know that every x in $\mathfrak{J}(\mathcal{A})$ can be written as a linear combination of some elements $x_1,x_2,\cdots,x_k$ in $\mathfrak{J}(\mathcal{A})$ such that $x_k=p_{i_1}p_{i_2}\cdots p_{i_k}$, where $p_{i_1},p_{i_2},\cdots,p_{i_k}$ are idempotents in $\mathcal{A}$. Since ϕ is bilinear, to show the theorem, it is sufficient to prove that

$$\phi(a,p_1p_2\cdots p_n)=\frac{1}{2}\phi(p_1p_2\cdots p_na,1)+\frac{1}{2}\phi(ap_1p_2\cdots p_n,1)\tag{2.1}$$

for all a and idempotents p_i in $\mathcal{A}$.