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On the Laplacian coefficients of signed graphs

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A R T I C L E I N F O

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A B S T R A C T

Let $\Gamma = (G, \sigma)$ be a signed graph, where G is its underlying graph and σ its sign function (defined on edges of G). A signed graph Γ', the subgraph of Γ, is its signed TU-subgraph if the signed graph induced by the vertices of Γ' consists of trees and/or unbalanced unicyclic signed graphs. Let $L(\Gamma) = D(G) - A(\Gamma)$ be the Laplacian of Γ. In this paper we express the coefficient of the Laplacian characteristic polynomial of Γ based on the signed TU-subgraphs of Γ, and establish the relation between the Laplacian characteristic polynomial of a signed graph with adjacency characteristic polynomials of its signed line graph and signed subdivision graph. As an application, we identify the signed unicyclic graphs having extremal coefficients of the Laplacian characteristic polynomial.

1. Introduction

Let $G = (V(G), E(G))$ be a graph of order $n = |V(G)|$ and size $m = |E(G)|$, and let $\sigma : E(G) \to \{+, -\}$ be a mapping defined on the edge set of G. Then $\Gamma = (G, \sigma)$ is a

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signed graph (or sigraph for short). The graph G is its underlying graph, while σ its sign function (or signature). Furthermore, it is common to interpret the signs as the integers $\{+1, -1\}$. Hence, sometimes signed graphs are treated as weighted graphs, whose (edge) weights belong to $\{1, -1\}$. An edge e is *positive* (*negative*) if $\sigma(e) = +$ (resp. $\sigma(e) = -$). If all edges in Γ are positive (negative), then Γ is denoted by $(G, +)$ (resp. $(G, -)$).

Most of the concepts defined for graphs are directly extended to signed graphs. For example, the degree of a vertex v in G (denoted by $deg(v)$) is also its degree in Γ. So $\Delta(G)$, the maximum (vertex) degree in G, also stands for $\Delta(\Gamma)$, interchangeably. Furthermore, if some subgraph of the underlying graph is observed, then the sign function for the subgraph is the restriction of the previous one. Thus, if $v \in V(G)$, then $\Gamma - v$ denotes the signed subgraph having $G-v$ as the underlying graph, while its signature is the restriction from $E(G)$ to $E(G-v)$ (note, all edges incident to v are deleted). If $U \subset V(G)$ then $\Gamma[U]$ or $G(U)$ denotes the (signed) induced subgraph arising from U, while $\Gamma - U = \Gamma[V(G)\backslash U]$. Sometimes we also write $\Gamma - \Gamma[U]$ instead of $\Gamma - U$. A cycle of Γ is said to be *balanced* (or *positive*) if it contains an even number of negative edges, otherwise it is *unbalanced* (or, *negative*). A signed graph is said to be *balanced* if all its cycles are balanced; otherwise, it is *unbalanced*. For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \backslash U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be (signature) *switching equivalent* to Γ. In fact, switching equivalent signed graphs can be considered as *switching isomorphic* graphs and their signatures are said to be equivalent. Observe also that switching equivalent graphs have the same set of positive cycles.

Given a (simple) graph G, then its *line graph* $\mathcal{L}(G)$ has as its vertex set the edge set of G, with two vertices in $\mathcal{L}(G)$ being adjacent whenever the corresponding edges of G have a common vertex; its *subdivision graph* $\mathcal{S}(G)$ is obtained from G by inserting into each edge of G a vertex of degree 2. If G is a signed graph, then the resulting graphs are signed graphs (to be defined later).

Simple graphs are widely studied in the literature by means of the eigenvalues of several matrices associated to graphs. Among them, the most common are the adjacency and Laplacian matrix. Given a graph G, $A(G) = (a_{ij})$ is its adjacency matrix if $a_{ij} = 1$ whenever vertices i and j are adjacent and $a_{ij} = 0$ otherwise; $L(G) = D(G) - A(G)$ is its Laplacian matrix, where $D(G)$ is the diagonal matrix of vertex degrees. Since recently, the so-called signless Laplacian matrix, defined as $Q(G) = A(G) + D(G)$, has attracted much attention in the literature, see for example [4,6–8]. For signed graphs we consider the analogous matrices. Let $e = ij$ be an edge of G joining vertices i and j. The adjacency matrix $A(\Gamma) = (a^\sigma_{ij})$ with $a^\sigma_{ij} = \sigma(ij)a_{ij}$ is called the signed adjacency matrix; $L(\Gamma) = D(G) - A(\Gamma)$ is the corresponding Laplacian matrix. Note, $L(G, +) = L(G)$, while $L(G, -) = Q(G)$.

In this paper we will consider both, the characteristic polynomial of the adjacency matrix and of the Laplacian matrix of a signed graph Γ. To avoid a confusion we denote by