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Computing the degree of a vertex in the skeleton of acyclic Birkhoff polytopes ☆

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For a fixed tree T with n vertices the corresponding acyclic Birkhoff polytope $\Omega_n(T)$ consists of doubly stochastic matrices having support in positions specified by T (matrices associated with T). The skeleton of $\Omega_n(T)$ is the graph whose vertices are the permutation matrices associated with T and two vertices (permutation matrices) A and B are adjacent if and only if $(E(G_A) \setminus E(G_B)) \cup (E(G_B) \setminus E(G_A))$ is the edge set of a nontrivial path, where $E(G_A)$ and $E(G_B)$ are the edge sets of graphs associated with A and B, respectively. We present a formula to compute the degree of any vertex in the skeleton of $\Omega_n(T)$. We also describe an algorithm for computing this number. In addition, we determine the maximum degree of a vertex in the skeleton of $\Omega_n(T)$, for certain classes of trees, including paths and generalized stars where the branches have equal length.

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1. Introduction

Let $T = (V(T), E(T))$ be a tree with vertex set $V(T) = \{v_1, \ldots, v_n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. For each k, $1 \leq k \leq n-1$, the edge $e_k = \{v_i, v_j\}$ is simply denoted by v_iv_j and we say that v_i is adjacent to v_j. The neighbors of v_i are its adjacent vertices. The degree of a vertex v_i is the number of its neighbors and is denoted by $d_T(v_i)$. An edge v_iv_j is a pendant edge when v_i or v_j has degree 1. A vertex v_i is s pendant vertex of the tree when v_i has degree 1. A vertex of T which is not a pendant vertex is called an inner vertex. A path with $n \geq 2$ vertices, P_n, is a tree in which $n-2$ vertices have degree 2 and the others have degree 1. Usually, the path P_n is denoted by its vertices. The length of P_n is its number of vertices. For more basic definitions and notations of trees, see [3].

A matching in T is a subset $M \subseteq E(T)$ without two edges adjacent in T (edge e_1 is adjacent to edge e_2 when they have a common vertex). Let M and M' be two matchings in T. A path P in T where edges alternate between being in M and M' is called an alternating path (relative to (M, M')). Similarly, a nontrivial path P in T where edges alternate being in M and not in M, is called an M-alternating path. We allow an M-alternating path to consist of a single edge, which is either in M or not in M. The symmetric difference of two sets F_1 and F_2 is $F_1 \Delta F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$.

Let $\mathbb{R}^{E(T)}$ be the vector space of functions from $E(T)$ into $\mathbb{R}$. We write $x \geq 0$ to indicate componentwise nonnegativity ($x_e \geq 0$ for each $e \in E(T)$). The matching polytope associated with T and denoted by $\mathcal{M}(T)$, is

$$\mathcal{M}(T) = \left\{ x \in \mathbb{R}^{E(T)};\ x \geq 0,\ \sum_{e \in E(v_i)} x_e \leq 1 \quad (i \leq n) \right\}.$$

For these concepts see [11,12].

The vertices of the skeleton, $G(\mathcal{M}(T))$, of $\mathcal{M}(T)$ are the vertices of $\mathcal{M}(T)$ and so, matchings in T. Using a result from [10] the authors of [1] showed that two vertices M and M' of $G(\mathcal{M}(T))$ are adjacent if and only if $M \Delta M'$ is a nontrivial alternating path.

A real n-by-n matrix is a doubly stochastic matrix if it is a nonnegative matrix and each row and column sum is 1, [4,5]. The set of all doubly stochastic matrices of order n is a polytope and is denoted by Ω_n. Polytopes and in particular Ω_n are studied in many papers [1,4–8,11]. Each permutation matrix is an extreme point (vertices) of Ω_n. The theorem of Birkhoff asserts that Ω_n has no other extreme points, [2,4,5].

The acyclic Birkhoff polytope, $\Omega_n(T)$, introduced in [6], is the set of doubly stochastic matrices A such that each positive entry of A is either on the diagonal or in a position that corresponds to an edge of T. The diagonal entries of A correspond to the vertices of T. Each matrix $A \in \Omega_n(T)$ is symmetric (see [6]). The acyclic Birkhoff polytope $\Omega_n(T)$ is affinely isomorphic (see [6]) to the matching polytope associated with T, $\mathcal{M}(T)$. So, the skeleton of the polytope $\Omega_n(T)$, denoted by $G(\Omega_n(T))$, is the graph whose vertices are the vertices of $\Omega_n(T)$ (permutation matrices in $\Omega_n(T)$). Two vertices A and B of