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Finite two-distance tight frames

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A finite collection of unit vectors $S \subset \mathbb{R}^n$ is called a spherical two-distance set if there are two numbers a and b such that the inner products of distinct vectors from S are either a or b. We prove that if $a \neq -b$, then a two-distance set that forms a tight frame for $\mathbb{R}^n$ is a spherical embedding of a strongly regular graph. We also describe all two-distance tight frames obtained from a given graph. Together with an earlier work by S. Waldron (2009) [22] on the equiangular case, this completely characterizes two-distance tight frames. As an intermediate result, we obtain a classification of all two-distance 2-designs.

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1. Introduction

A finite collection of unit vectors $S \subset \mathbb{R}^n$ is called a spherical two-distance set if there are two numbers a and b such that the inner products of distinct vectors from S are either a or b. If in addition $a = -b$, then S defines a set of equiangular lines through the origin in $\mathbb{R}^n$. Equiangular lines form a classical subject in discrete geometry following foundational papers of Van Lint, Seidel, and Lemmens [16,15]. Equiangular line sets are closely related to strongly regular graphs and two-graphs [9,10] which form the main source of their constructions. Another group of results is concerned with bounding the maximum size $g(n)$ of spherical two-distance sets in n dimensions. We refer to [3,4] for the latest results on upper bounds on $g(n)$ as well as an overview of the relevant literature.

A finite collection of vectors $S = \{x_1, \dots, x_N\} \subset \mathbb{R}^n$ is called a *finite frame* for the Euclidean space $\mathbb{R}^n$ if there are constants $0 < A \le B < \infty$ such that for all $x \in \mathbb{R}^n$

$$A||x||^2 \le \sum_{i=1}^{N} \langle x, x_i \rangle^2 \le B||x||^2. \tag{1.1}$$

If $A = B$, then S is called an A-tight frame, in which case

$$A = \frac{1}{n} \sum_i \|x_i\|^2. \tag{1.2}$$

It is trivially seen that a finite collection of vectors $S = \{x_i : i = 1, \dots N\} \subset \mathbb{R}^n$ is an A-tight frame if and only if for any $x \in \mathbb{R}^n$,

$$Ax = \sum_{i=1}^{N} \langle x, x_i \rangle x_i. \tag{1.3}$$

If in addition $\|x_i\| = 1$ for all $i \in I$, then S is a finite unit-norm tight frame or FUNTF. If at the same time S is a spherical two-distance set, we call it a *two-distance tight frame*. In particular, if the two inner products in S satisfy the condition $a = -b$, then it is an *equiangular tight frame* or ETF. All frames in this paper will be assumed unit-norm.

The Gram matrix G of S is defined by $G_{ij} = \langle x_i, x_j \rangle, 1 \le i, j \le N$, where $N = |S|$. If S is a FUNTF for $\mathbb{R}^n$, then it is straightforward to show [11] that G has one nonzero eigenvalue $\lambda = N/n$ of multiplicity n and an eigenvalue 0 of multiplicity $N - n$.

Frames have been used in signal processing and have a large number of applications in sampling theory, wavelet theory, data transmission, and filter banks [7,13]. The study of ETFs was initiated by Strohmer and Heath [21] and Holmes and Paulsen [12]. In

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