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Maps on matrices compressing the local spectrum in the spectrum

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ABSTRACT

Let $M_n(\mathbb{C})$ be the algebra of all complex $n \times n$ matrices. Let x_0 be a nonzero vector in $\mathbb{C}^n$. We describe maps ϕ on $M_n(\mathbb{C})$ (not linear or surjective) satisfying

$$\sigma_{T \pm S}(x_0) \subseteq \sigma(\phi(T) \pm \phi(S))$$

for all $T, S \in M_n(\mathbb{C})$. We also obtain a similar description by supposing that ϕ is surjective and $\sigma_{\phi(T) \pm \phi(S)}(x_0) \subseteq \sigma(T \pm S)$ for all $T, S \in M_n(\mathbb{C})$.

1. Introduction and statement of the main results

Let $\mathcal{B}(X)$ be the algebra of all linear bounded operators on a complex Banach space X. The local resolvent set of an operator $T \in \mathcal{B}(X)$ at a point $x \in X$, denoted by $\rho_T(x)$, is the union of all open subsets $U \subseteq \mathbb{C}$ for which there exists an analytic function $f : U \to X$ such that $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. The local spectrum of T at x is defined by

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$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ and is a (possibly empty) closed subset of $\sigma(T)$, the usual spectrum of $T \in \mathcal{B}(X)$.

The problem of characterizing spectrum-preserving maps on the algebra $M_n(\mathbb{C})$ of all complex $n \times n$ matrices was considered by a number of authors. In [19], Marcus and Moyls proved that if a linear map ϕ on $M_n(\mathbb{C})$ preserves eigenvalues (counting multiplicities), then there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that

$$\phi(T) = ATA^{-1}, \ (T \in M_n(\mathbb{C})), \tag{1}$$

or

$$\phi(T) = AT^tA^{-1}, \ (T \in M_n(\mathbb{C})), \tag{2}$$

where T^t denotes, as usual, the transpose of $T \in M_n(\mathbb{C})$. This result has been generalized in different directions; see [4,12,14,20]. In particular, it is shown in [12,14] that for a map ϕ on $M_n(\mathbb{C})$ with $\phi(0) = 0$, the following statements are equivalent.

1. ϕ satisfies

$$\sigma(\phi(T) - \phi(S)) \subseteq \sigma(T - S), \ (T, S \in M_n(\mathbb{C})), \tag{3}$$

2. ϕ satisfies

$$\sigma(T - S) \subseteq \sigma(\phi(T) - \phi(S)), \ (T, S \in M_n(\mathbb{C})), \tag{4}$$

3. ϕ takes either the form (1) or (2).

This result has been shown before under additional assumptions such as continuity or surjectivity of the map ϕ. But as stated above, the equivalence between the last two statements is due to Costara [12] and the equivalence between the first and third statements was recently proved by Dolinar, Hou, Kuzma and Qi in [14].

The problem of describing linear or additive maps on $\mathcal{B}(X)$ preserving the local spectra has been initiated by Bourhim and Ransford in [8], and continued by several authors; see for instance [5–7,10,11,13,16,17]. In [4], Bendaoud, Douimi and Sarih used Costara's approach and proved, for a fixed nonzero vector x_0 in $\mathbb{C}^n$, that a map ϕ on $M_n(\mathbb{C})$ with $\phi(0) = 0$ satisfies

$$\sigma_{\phi(T)-\phi(S)}(x_0) \subseteq \sigma_{T-S}(x_0), \ (T, S \in M_n(\mathbb{C})) \tag{5}$$

if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\phi(T) = ATA^{-1}$ for all $T \in M_n(\mathbb{C})$. They also show that the same conclusion remains valid when the reverse set inclusion in (5) occurs without surjectivity assumption on ϕ.