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Interpolated inequalities for unitarily invariant norms

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ABSTRACT

In this article we interpolate the well known Young and Heinz inequalities for unitarily invariant norms, and some of their known refinements. Then we prove new interpolated refinements. In the end, we use this interpolation idea to prove a hidden monotonicity behavior these inequalities obey.

1. Introduction

The ν-version of Young's inequality

$$a^\nu b^{1-\nu} \le \nu a + (1-\nu)b; a, b > 0, 0 \le \nu \le 1 \tag{1.1}$$

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has attracted many researchers due to its applications in the theory of inequalities. This inequality can be written in different equivalent ways, among which is the (p, q)-version

$$a^p b^q \leq \frac{p}{p+q} a^{p+q} + \frac{q}{p+q} b^{p+q}; a, b, p, q > 0. \tag{1.2}$$

This inequality has been studied, refined and extended to other spaces, and one of the most interesting applications is the matrix versions. In the sequel, $\mathbb{M}_n$ will denote the set of all $n \times n$ complex matrices, while $\mathbb{M}_n^+$ will denote the set of positive semidefinite matrices in $\mathbb{M}_n$. The ν-version of Young's inequality for matrices states that

$$|||A^\nu X B^{1-\nu}||| \leq \nu |||AX||| + (1-\nu)|||XB|||; A, B \in \mathbb{M}_n^+, X \in \mathbb{M}_n, 0 \leq \nu \leq 1, \tag{1.3}$$

where $||| \ |||$ is any unitarily invariant norm. The first proof of this inequality is due to [1], where a stronger version was given for the singular values.

Since then, the inequality has been studied extensively in different approaches. For example, in [5] the following refinement of the ν-version of real numbers was proved

$$(a^\nu b^{1-\nu} + a^{1-\nu} b^\nu)^2 + 2r_0(a-b)^2 \leq (a+b)^2, \tag{1.4}$$

where $a, b > 0, 0 \leq \nu \leq 1$ and $r_0 = \min\{\nu, 1-\nu\}$. In fact the proof of this refinement was based on the other refinement

$$a^\nu b^{1-\nu} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \nu a + (1-\nu)b,$$

proved in the same paper. On the other hand, the refinement

$$(a^\nu b^{1-\nu})^2 + r_0^2 (a-b)^2 \leq (\nu a + (1-\nu)b)^2 \tag{1.5}$$

was proved earlier in [4].

The study of this inequality is wide, hence we refer the reader to [2,5,6], to mention a few, for more on these and related inequalities.

In fact, one can say that the study of these inequalities is targeting matrices more than real numbers. It can be seen that each of the aforementioned inequalities has its own matrix version, with the use of some norm. For example, the matrix version of (1.5) is

$$\|A^\nu X B^{1-\nu}\|_2^2 + r_0^2 \|AX - XB\|_2^2 \leq \|\nu AX + (1-\nu)XB\|_2,$$

where $A, B \in \mathbb{M}_n^+$, $X \in \mathbb{M}_n$ and $\| \ \|_2$ is the Hilbert–Schmidt norm. Again we refer the reader to the references to see more examples on such matrix versions.

Our main goal in this paper is to interpolate the (p, q)-version of Young's inequality for unitarily invariant norms, then to interpolate some of the known refinements and to give new interpolated refinements. In the end, these interpolations will be used to study some monotonicity behavior of these inequalities.