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On the inversion of infinite moment matrices

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ABSTRACT

Motivated by [8] we study the existence of the inverse of an infinite Hermitian positive definite matrix (in short, HPD matrix) from the point of view of the asymptotic behaviour of the smallest eigenvalues of the finite sections. We prove a sufficient condition to assure the inversion of an HPD matrix with square summable rows. For infinite Toeplitz matrices we introduce the notion of asymptotic Toeplitz matrix and we show that, under certain assumptions, the inverse of an infinite Toeplitz positive definite matrix is asymptotic Toeplitz. Such inverses are computed in terms of the limits of the coefficients of the associated orthogonal polynomials. We apply these results in the context of the theory of orthogonal polynomials. In particular, we show that for measures on the unit circle $\mathbb{T}$ verifying that the smallest eigenvalue of the finite sections of the corresponding moment matrix are away from zero in the limit we may assure the existence of all the limits of the coefficients of the orthonormal polynomials with respect to such measures.

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1. Introduction

Let $\mathbf{M} = (c_{i,j})_{i,j=0}^{\infty}$ be an infinite Hermitian matrix, i.e., $c_{i,j} = \overline{c_{j,i}}$ for all i, j non-negative integers. Following [12] we say that $\mathbf{M}$ is positive definite (in short, an HPD matrix) if $|\mathbf{M}_n| > 0$ for all $n \geq 0$, where $\mathbf{M}_n$ is the truncated matrix of size $(n+1)\times(n+1)$ of $\mathbf{M}$. To each HPD matrix $\mathbf{M}$ can be associated an inner product on the linear space of polynomials $\mathbb{P}[z]$ as follows: if $p(z) = \sum_n v_n z^n$ and $q(z) = \sum_n w_n z^n$ then

$$\langle p, q\rangle = \begin{pmatrix} v_0 & v_1 & v_2 & \dots \end{pmatrix} \mathbf{M} \begin{pmatrix} \overline{w}_0 \\ \overline{w}_1 \\ \overline{w}_2 \\ \vdots \end{pmatrix}. \tag{1}$$

$\mathbf{M}$ is the Gram matrix of the inner product (1) in the vector space of polynomials $\mathbb{P}[z]$, i.e., $< z^i, z^j >= c_{i,j}$. Let $\{P_n(z)\}_{n=0}^{\infty}$ be the orthonormal polynomials with respect to such inner product and write

$$P_n(z) = \sum_{k=0}^{n} b_{k,n} z^k.$$

The orthonormal polynomials $\{P_n(z)\}_{n=0}^{\infty}$ are uniquely determined by orthonormality if $b_{n,n} > 0$ and can be given as (see e.g. [19])

$$P_n(z) = \frac{1}{\sqrt{|M_{n-1}||M_n|}} \begin{vmatrix} \overline{c_{0,0}} & \overline{c_{0,1}} & \dots & \overline{c_{0,n}} \\ \overline{c_{1,0}} & \overline{c_{1,1}} & \dots & \overline{c_{1,n}} \\ \vdots & \vdots & & \vdots \\ \overline{c_{n-1,0}} & \overline{c_{n-1,1}} & \dots & \overline{c_{n-1,n}} \\ 1 & z & \dots & z^n \end{vmatrix}, n \geq 1, \quad P_0 = 1. \tag{2}$$

Define the infinite upper triangular matrix $\mathbf{B} = (b_{k,n})_{k,n=0}^{\infty}$ with $b_{k,n} = 0$ if $k > n$. This matrix can be considered as the infinite transition matrix from the algebraic basis $\mathfrak{B}' = \{P_n(z)\}_{n=0}^{\infty}$ to $\mathfrak{B} = \{z^n\}_{n=0}^{\infty}$. Indeed, each finite section $\mathbf{B}_n$ of $\mathbf{B}$ is the corresponding transition matrix from the algebraic basis $\mathfrak{B}'_n = \{P_0(z), \dots, P_n(z)\}$ in the linear vector space $\mathbb{P}_n[z]$ of polynomials of degree at most n, to the standard basis $\mathfrak{B}_n = \{1, z, \dots, z^n\}$. Since $\mathbf{M}_n$, $\mathbf{I}_n$ are both matricial representations of the same inner product with respect to $\mathfrak{B}_n$, $\mathfrak{B}'_n$ respectively, then

$$\mathbf{B}_n^t \mathbf{M}_n \overline{\mathbf{B}}_n = \mathbf{I}_n,$$

and consequently $\mathbf{M}_n^{-1} = \overline{\mathbf{B}_n} \mathbf{B}_n^t$. This result in the case of Hankel matrices going back to A.C. Aitken, cf. [10] has been recovered several times see [5,8] (also [12] in the context of moment Hermitian matrices). In particular, if $\mathbf{A}_n = \overline{\mathbf{B}_n} \mathbf{B}_n^t$, it follows that