Linear Algebra and its Applications 475 (2015) 306–325

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Kneading determinants of infinite order linear recurrences

João F. Alves [a,*], António Bravo [b], Henrique M. Oliveira [a]

[a] Centro de Análise Matemática Geometria e Sistemas Dinâmicos, Math. Dep., Tech. Institute of Lisbon, Univ. of Lisbon, Av. Rovisco Pais, 1049-001 Lisbon, Portugal

[b] Centro de Análise Funcional e Aplicações, Math. Dep., Tech. Institute of Lisbon, Univ. of Lisbon, Av. Rovisco Pais, 1049-001 Lisbon, Portugal

ARTICLE INFO

Article history:
Received 9 July 2013
Accepted 27 February 2015
Available online 10 March 2015
Submitted by R. Brualdi

MSC:
15A15
39A06

Keywords:
Kneading matrix
Kneading determinant
Infinite linear recurrence
Fibonacci recurrence
Binet Formula
Infinite matrices

ABSTRACT

Infinite order linear recurrences are studied via kneading matrices and kneading determinants. The concepts of kneading matrix and kneading determinant of an infinite order linear recurrence, introduced in this work, are defined in a purely linear algebraic context. These concepts extend the classical notions of Frobenius companion matrix to infinite order linear recurrences and to the associated discriminant of finite order linear recurrences. Asymptotic Binet formulas are deduced for general classes of infinite order linear recurrences as a consequence of the analytical properties of the generating functions obtained for the solutions of these infinite order linear recurrences.

* Corresponding author.
E-mail addresses: jalves@math.tecnico.ulisboa.pt (J.F. Alves), abravo@math.tecnico.ulisboa.pt (A. Bravo), holiv@math.tecnico.ulisboa.pt (H.M. Oliveira).

http://dx.doi.org/10.1016/j.laa.2015.02.035

1. Introduction

The concept of kneading determinant was introduced by Milnor and Thurston [7] in the late eighties of the last century in the context of one-dimensional dynamics. Later on, it was shown that the kneading determinant of an interval map can be regarded as the determinant of a pair of linear endomorphisms with finite rank, see [1] and [2]. This latter point of view, purely linear algebraic, is the link between the Milnor and Thurston notion and our definition of kneading determinant of a linear recurrence. Indeed, as we will see, the kneading determinant of an infinite order linear recurrence is a particular case of the above mentioned determinant of a pair of linear endomorphisms with finite rank described in [2].

The main objective of this paper is to show that the kneading determinants play an important role in the study of infinite vector recurrences, giving directly the generating functions of the solution of the problem. In addition, the determinants present a powerful computational tool to obtain the actual solutions of finite and infinite order linear recurrences.

Linear recurrences have a long history, they constitute generalizations of the eight centuries old finite linear recurrences of Leonardo de Pisa, or Fibonacci [10]

$$q_{n+1} = q_n + q_{n-1}, \text{ with } q_0 = 0,\ q_1 = 1 \text{ and } n \geq 1.$$

In the 19th century Jacques Philippe Marie Binet popularized a formula, earlier known to De Moivre, solving the Fibonacci recurrence as a function of n.

In a series of papers [8,11,3], Rachidi and other authors studied linear infinite order scalar recurrences. Given an infinite sequence of coefficients $\{a_i\}_{i=0,1,2\dots}$, with some possible conditions on the sequence, like periodicity [3], positivity of the coefficients, see the recent work [9], or the existence of some limit, the problem was to find a solution of the infinite order linear scalar recurrences

$$q_{n+1} = \sum_{i=0}^{+\infty} a_i q_{n-i}, \text{ for } n \geq 0, \tag{1}$$

with an infinite set of initial conditions $\{q_i\}_{i=0,-1,-2,\dots}$. By studying the results of these researchers, namely on Binet formulas, we adopted a new approach to the problem using the different technique of kneading determinants. We apply this new method to a wider class of recurrences, obtaining solutions and asymptotic behaviour showing the conceptual and computational power of kneading determinants. One of the advantages of using generating functions is the possibility of analyzing the asymptotic behaviour using the analytical properties of the generating function.

The paper is organized as follows, in Sections 2 and 3, we introduce the terminology and the main results of this paper, we generalize this problem to vectorial recurrences (2) and present their solutions. Naturally, the method solves scalar recurrences as a particular case. We present three fundamental results characterizing the solutions of infinite