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Periodic harmonic functions on lattices and Chebyshev polynomials

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A R T I C L E   I N F O

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A B S T R A C T

We shall give an explicit expression of the dimension of the space of harmonic functions on the Cartesian product of path (resp. cycle) graphs in terms of Chebyshev polynomials of the second (resp. first) kind. As an application, we obtain several identities among the dimensions, some are new and some are known but obtained previously by other methods. Our motivation for this study is the "Lights Out" puzzle.

1. Introduction

Let $G = (V, E)$ be a finite undirected simple graph and K an arbitrary field. A function on V with values in K is called a configuration. Let $C_{G,K}$ denote the set of all configurations. It is regarded as a vector space over K. For $a \in K$, we define the endomorphism $\Delta_{G,K,a}$ of $C_{G,K}$, which we call the a-Laplacian, by

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$$\Delta_{G,K,a}(f)(v) := af(v) + \sum_{(u,v)\in E} f(u).$$

In the case where G is r-regular, the ordinary Laplacian is $-\Delta_{G,K,-r}$. We are interested in the dimension of the space of "a-harmonic functions"

$$d(G, K, a) := \dim_K \ker \Delta_{G,K,a}.$$

Let $\mathbf{P}_n$ denote the path graph with n vertices ($n \geq 2$) and $\mathbf{C}_n$ the cycle graph with n vertices ($n \geq 3$). Let $G \times H$ denote the Cartesian product of graphs G and H. The number $d(\mathbf{P}_n \times \mathbf{P}_n, \mathbb{F}_2, 1)$ (resp. $d(\mathbf{C}_n \times \mathbf{C}_n, \mathbb{F}_2, 1)$) has attracted special attention in connection with the "Lights Out" puzzle (resp. the torus version of this puzzle); see the references [1–4,8–13]. The behavior of these numbers is rather mysterious; see [3, Table 1] for the values of $d(\mathbf{C}_n \times \mathbf{C}_n, \mathbb{F}_2, 1)$, $n \leq 300$.

In this paper we shall give an explicit expression of $d(\mathbf{P}_m \times \mathbf{P}_n, K, a)$ and $d(\mathbf{C}_m \times \mathbf{C}_n, K, a)$ in terms of Chebyshev polynomials of the second and first kind, respectively. A configuration for $\mathbf{C}_m \times \mathbf{C}_n$ is naturally identified with a function on $\mathbb{Z}^2$ which is (m, n)-periodic, hence the title of this paper. The normalized Chebyshev polynomials of the first and the second kind are defined by

$$C_0(x) = 2, \qquad C_1(x) = x, \qquad C_n(x) = xC_{n-1}(x) - C_{n-2}(x)\ (n \geq 2),$$
$$S_0(x) = 1, \qquad S_1(x) = x, \qquad S_n(x) = xS_{n-1}(x) - S_{n-2}(x)\ (n \geq 2),$$

respectively. We put $\widetilde{C}_n(x) := C_n(x) - 2$. Let $\operatorname{ord}_p(n)$ denote the p-adic additive valuation of n. The main result is the following.

Theorem 1.1.

(i) $d(\mathbf{P}_m \times \mathbf{P}_n, K, a) = \deg \gcd{}_K(S_m(x), S_n(-x - a))$.
(ii) $d(\mathbf{C}_m \times \mathbf{C}_n, K, a) = 2 \deg \gcd{}_K(\widetilde{C}_m(x), \widetilde{C}_n(-x - a)) - \varepsilon$, *where*
- $\varepsilon = 2$ *if* $\operatorname{char} K = p \geq 3$, $a = 0$, $\operatorname{ord}_p(m) = \operatorname{ord}_p(n)$, *and both* m, n *are even,*
- $\varepsilon = 1$ *if either*
  - $\operatorname{char} K = 2$, $a = 0$, *and both* m, n *are odd,*
  - $\operatorname{char} K = p \geq 3$, $a = -4$, $\operatorname{ord}_p(m) = \operatorname{ord}_p(n)$,
  - $\operatorname{char} K = p \geq 3$, $a = 4$, $\operatorname{ord}_p(m) = \operatorname{ord}_p(n)$, *and both* m, n *are even, or*
  - $\operatorname{char} K = p \geq 3$, $a = 0$, $\operatorname{ord}_p(m) = \operatorname{ord}_p(n)$, *and either* m *or* n *is even,*
- $\varepsilon = 0$ *otherwise.*

Theorem 1.1 (i) was essentially known in the case $\operatorname{char} K = 2$, $a = 0$ (cf. Remark 4.2). In the case $a = 1$, (i) was proved in [10] (see also [4]). The equality (ii) for $a = 1$ was conjectured in [3] in the case $\operatorname{char} K = 2$ and in [11] in the case $\operatorname{char} K = p > 0$.

The organization of this paper is as follows. Basic properties of Chebyshev polynomials are gathered in Section 2. It should be pointed out that Chebyshev polynomials of the