

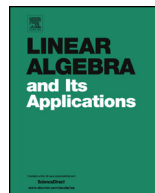


ELSEVIER

Contents lists available at ScienceDirect

## Linear Algebra and its Applications

www.elsevier.com/locate/laa



## Quadratic homogeneous Keller maps of rank two



Kevin Pate, Charles Ching-An Cheng\*

Department of Mathematics and Statistics, Oakland University, Rochester, MI,  
United States

## ARTICLE INFO

## Article history:

Received 13 June 2014

Accepted 6 February 2015

Available online 10 March 2015

Submitted by J.M. Landsberg

## MSC:

14R10

14R20

## Keywords:

Quadratic

Homogeneous

Polynomial map

Keller map

Nilpotent

Strongly nilpotent

Linearly triangularizable

## ABSTRACT

Let  $H$  be a quadratic homogeneous polynomial map of dimension  $n$  over an infinite field in which 2 is invertible such that its Jacobian  $JH$  is nilpotent. Meisters and Olech have shown that  $JH$  is strongly nilpotent if  $n \leq 4$ . They also proved that it is not true when  $n = 5$ . We show that if  $\text{rank } JH \leq 2$  and  $n$  arbitrary, then  $JH$  is strongly nilpotent. We also give examples to show that this is no longer true for any rank and dimension as long as the rank is greater than 2.

© 2015 Elsevier Inc. All rights reserved.

## 1. Introduction

Let  $k$  be a field and  $H = (H_1, \dots, H_n) : k^n \rightarrow k^n$  a polynomial map, i.e.  $H(x) = (H_1(x), \dots, H_n(x))$  where  $H_i \in k[X] = k[X_1, \dots, X_n]$ . The degree of  $H$  is defined by  $\deg H = \max\{\deg H_i \mid i = 1, 2, \dots, n\}$ .  $H$  is *homogeneous* of degree  $d$  if each  $H_i$  is either homogeneous of degree  $d$  or equal to zero. A matrix  $M \in M_n(k[X_1, \dots, X_s])$ , is *strongly*

\* Corresponding author.

E-mail address: [cheng@oakland.edu](mailto:cheng@oakland.edu) (C.C.-A. Cheng).

nilpotent if  $M(v_1)M(v_2)\cdots M(v_n) = 0$  for any  $v_1, v_2, \dots, v_n \in k^s$ . Meisters and Olech [8] proved for any quadratic homogeneous map  $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that if  $JH$  is nilpotent then  $JH$  is strongly nilpotent if  $n \leq 4$  and the statement is false if  $n \geq 5$ .

In this paper we prove the following.

**Theorem 1.** *Suppose  $k$  is an infinite field with 2 invertible and  $H : k^n \rightarrow k^n$  is a quadratic homogeneous polynomial map such that  $JH$  is nilpotent. If  $JH \in M_n(k(X))$  has rank at most 2 then  $JH$  is strongly nilpotent.*

This extends the result of Meisters and Olech [8] when  $n = 3$ . When the field  $k$  has characteristic 0, an anonymous referee has pointed out that this result can be derived from [3] and a proof is given in Appendix A.

A polynomial map  $F$  is *invertible* if there exists a polynomial map  $G$  such that  $FG = GF = (X_1, \dots, X_n)$ . The map  $F$  is *Keller* if the determinant of its Jacobian,  $JF = (\partial F_i / \partial X_j)$ , is a nonzero element of  $k$ . By the chain rule for Jacobians, invertible polynomial maps are Keller maps. The famous Jacobian Conjecture states that if  $k$  has characteristic 0 then any Keller map is invertible. Wang [11] proved it if  $\deg F \leq 2$ . Bass, Connell and Wright [1] reduced the conjecture to the case where  $F = X + H$  with  $H$  homogeneous of degree 3. (Druzkowski [5] has further reduced it to where  $H$  is cubic linear.) When  $H$  is homogeneous of degree  $\geq 2$ , it can be shown that  $F = X + H$  is Keller if and only if  $JH$  is nilpotent (see [1] or [12]).

A polynomial map of the form  $(X_1 + p_1, \dots, X_n + p_n)$  is *lower triangular* if each  $p_i$  is a polynomial in  $k[X_1, \dots, X_{i-1}]$ . It is *upper triangular* if each  $p_i$  is a polynomial in  $k[X_{i+1}, X_{i+2}, \dots, X_n]$ . An upper triangular map can be turned into a lower triangular map, and vice versa, by conjugating it with the invertible linear map  $(X_n, \dots, X_1)$ . A polynomial map  $F$  is *triangular* if it is either upper or lower triangular. It is *linearly triangularizable* (LT) if it is linearly conjugate to a triangular map, i.e., there exists an invertible linear map  $T$  such that  $T^{-1}FT$  is triangular.

When  $k$  has characteristic 0, van den Essen and Hubbers [7] proved that  $JH$  is strongly nilpotent if and only if the polynomial map  $X + H$  is LT. Thus we may deduce from Theorem 1 the following.

**Corollary 2.** *Suppose  $k$  has characteristic 0 and  $F = X + H$  is a quadratic homogeneous Keller polynomial map with  $\text{rank } JH \leq 2$ . Then  $F$  is LT.*

This extends a result of Cheng [4] that all quadratic linear maps of rank 2 are LT.

In Section 2 we prove Theorem 1. In Section 3 we prove a technical lemma which is needed for the proof of Theorem 1. In Section 4 we exhibit examples to show that the conclusion of Theorem 1 is no longer true if the rank of  $JH$  is greater than 2. In Appendix A we provide another proof of Theorem 1 when the field  $k$  is algebraically closed with characteristic 0.

Download English Version:

<https://daneshyari.com/en/article/4599108>

Download Persian Version:

<https://daneshyari.com/article/4599108>

[Daneshyari.com](https://daneshyari.com)