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Quadratic homogeneous Keller maps of rank two

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ABSTRACT

Let H be a quadratic homogeneous polynomial map of dimension n over an infinite field in which 2 is invertible such that its Jacobian JH is nilpotent. Meisters and Olech have shown that JH is strongly nilpotent if $n \leq 4$. They also proved that it is not true when $n = 5$. We show that if rank $JH \leq 2$ and n arbitrary, then JH is strongly nilpotent. We also give examples to show that this is no longer true for any rank and dimension as long as the rank is greater than 2.

1. Introduction

Let k be a field and $H = (H_1, \ldots, H_n) : k^n \to k^n$ a polynomial map, i.e. $H(x) = (H_1(x), \ldots, H_n(x))$ where $H_i \in k[X] = k[X_1, \ldots, X_n]$. The *degree* of H is defined by $\deg H = \max\{\deg H_i | i = 1, 2, \ldots, n\}$. H is *homogeneous* of degree d if each H_i is either homogeneous of degree d or equal to zero. A matrix $M \in M_n(k[X_1, \ldots, X_s])$, is *strongly*

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nilpotent if $M(v_1)M(v_2)\cdots M(v_n) = 0$ for any $v_1, v_2, \ldots, v_n \in k^s$. Meisters and Olech [8] proved for any quadratic homogeneous map $H : \mathbb{R}^n \to \mathbb{R}^n$ that if JH is nilpotent then JH is strongly nilpotent if $n \leq 4$ and the statement is false if $n \geq 5$.

In this paper we prove the following.

Theorem 1. *Suppose k is an infinite field with 2 invertible and $H : k^n \to k^n$ is a quadratic homogeneous polynomial map such that JH is nilpotent. If $JH \in \mathrm{M}_n(k(X))$ has rank at most 2 then JH is strongly nilpotent.*

This extends the result of Meisters and Olech [8] when $n = 3$. When the field k has characteristic 0, an anonymous referee has pointed out that this result can be derived from [3] and a proof is given in Appendix A.

A polynomial map F is *invertible* if there exists a polynomial map G such that $FG = GF = (X_1, \ldots, X_n)$. The map F is *Keller* if the determinant of its Jacobian, $JF = (\partial F_i/\partial X_j)$, is a nonzero element of k. By the chain rule for Jacobians, invertible polynomial maps are Keller maps. The famous Jacobian Conjecture states that if k has characteristic 0 then any Keller map is invertible. Wang [11] proved it if $\deg F \leq 2$. Bass, Connel and Wright [1] reduced the conjecture to the case where $F = X + H$ with H homogeneous of degree 3. (Druzkowski [5] has further reduced it to where H is cubic linear.) When H is homogeneous of degree ≥ 2, it can be shown that $F = X + H$ is Keller if and only if JH is nilpotent (see [1] or [12]).

A polynomial map of the form $(X_1 + p_1, \ldots, X_n + p_n)$ is *lower triangular* if each p_i is a polynomial in $k[X_1, \ldots, X_{i-1}]$. It is *upper triangular* if each p_i is a polynomial in $k[X_{i+1}, X_{i+2}, \ldots, X_n]$. An upper triangular map can be turned into a lower triangular map, and vice versa, by conjugating it with the invertible linear map $(X_n, \ldots, X_1)$. A polynomial map F is *triangular* if it is either upper or lower triangular. It is *linearly triangularizable* (LT) if it is linearly conjugate to a triangular map, i.e., there exists an invertible linear map T such that $T^{-1}FT$ is triangular.

When k has characteristic 0, van den Essen and Hubbers [7] proved that JH is strongly nilpotent if and only if the polynomial map $X + H$ is LT. Thus we may deduce from Theorem 1 the following.

Corollary 2. *Suppose k has characteristic 0 and $F = X + H$ is a quadratic homogeneous Keller polynomial map with $\operatorname{rank} JH \leq 2$. Then F is LT.*

This extends a result of Cheng [4] that all quadratic linear maps of rank 2 are LT.

In Section 2 we prove Theorem 1. In Section 3 we prove a technical lemma which is needed for the proof of Theorem 1. In Section 4 we exhibit examples to show that the conclusion of Theorem 1 is no longer true if the rank of JH is greater than 2. In Appendix A we provide another proof of Theorem 1 when the field k is algebraically closed with characteristic 0.