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On traces of tensor representations of diagrams

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ABSTRACT

Let T be an (abstract) set of *types*, and let $\iota, o : T \to \mathbb{Z}_+$. A *$T$-diagram* is a locally ordered directed graph G equipped with a function $\tau : V(G) \to T$ such that each vertex v of G has indegree $\iota(\tau(v))$ and outdegree $o(\tau(v))$. (A directed graph is *locally ordered* if at each vertex v, linear orders of the edges entering v and of the edges leaving v are specified.)

Let V be a finite-dimensional $\mathbb{F}$-linear space, where $\mathbb{F}$ is an algebraically closed field of characteristic 0. A function R on T assigning to each $t \in T$ a tensor $R(t) \in V^{*\otimes\iota(t)} \otimes V^{\otimes o(t)}$ is called a *tensor representation* of T. The *trace* (or *partition function*) of R is the $\mathbb{F}$-valued function p_R on the collection of T-diagrams obtained by 'decorating' each vertex v of a T-diagram G with the tensor $R(\tau(v))$, and contracting tensors along each edge of G, while respecting the order of the edges entering v and leaving v. In this way we obtain a *tensor network*.

We characterize which functions on T-diagrams are traces, and show that each trace comes from a unique 'strongly nondegenerate' tensor representation. The theorem applies to virtual knot diagrams, chord diagrams, and group representations.

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1. Introduction

Our theorem characterizes traces of tensor networks, more precisely of tensor representations of diagrams, which applies to knot diagrams, group representations, and algebras. Tensor networks root in works of Cayley [4] and Clebsch [6], and their diagrammatical notation was pursued by Buchheim [3], Clifford [7], Sylvester [27], and Kempe [16,17]. They were revived by Penrose [23] and applied to knot theory by Kauffman [13] and to Hopf algebra in 'Kuperberg's notation' [18]. Other applications were found in areas like quantum complexity (cf. [1,12,21,25]), statistical physics (cf. [11,26]), and neural networks (cf. [22]). (See Landsberg [19] for an in-depth survey of the geometry of tensors and its applications.)

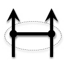
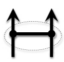
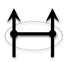
1.1. Types and T-diagrams

Let T be an (abstract, finite or infinite) set, the elements of which we call *types*, and let $\iota, o : T \to \mathbb{Z}_+$ (:= set of nonnegative integers). A *T-diagram* is a (finite) locally ordered directed graph G equipped with a function $\tau : V(G) \to T$ such that each vertex v of G has indegree $\iota(\tau(v))$ and outdegree $o(\tau(v))$. Here a directed graph is *locally ordered* if at each vertex v, a linear order of the edges entering v and a linear order of the edges leaving v are specified. Loops and multiple edges are allowed. Moreover, we allow the 'vertexless directed loop'  — more precisely, components of a T-diagram may be vertexless directed loops.

Let $\mathcal{G}_T$ denote the collection of all T-diagrams. If T is clear from the context, we call a T-diagram just a *diagram*, and denote $\mathcal{G}_T$ by $\mathcal{G}$. The types can be visualized by small pictograms indicating the type of any vertex, as in the following examples.

1.2. Examples

Virtual link diagrams. $T = \{$  ,  $\}$. So $|T| = 2$ and $\iota(t) = o(t) = 2$ for each $t \in T$. (In pictures like this we assume the entering edges are ordered counter-clockwise and the leaving edges are ordered clockwise. We also will occasionally delete the grey circle indicating the vertex.) Then the T-diagrams are the virtual link diagrams (cf. [14, 15,20]).

Multiloop chord diagrams. $T = \{$  $\}$, with $\iota($  $) = o($  $) = 2$. Then the T-diagrams are the multiloop chord diagrams, which play a key role in the Vassiliev knot invariants (cf. [5]). They can also be described as cubic graphs in which a set of disjoint oriented circuits ('Wilson loops') covering all vertices is specified. By contracting each Wilson loop to one point, the T-diagrams correspond to graphs cellularly embedded on an oriented surface.

Groups. Let Γ be a group, and let $T := \Gamma$, with $\iota(t) = o(t) = 1$ for each $t \in T$. Then T-diagrams consist of disjoint directed cycles, with each vertex typed by an element of Γ.