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Tight frames for cyclotomic fields and other rational vector spaces

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ABSTRACT

Here we consider the construction of tight frames for rational vector spaces. This is a subtle question, because the inner products on $\mathbb{Q}^d$ are not all isomorphic. We show that a tight frame for $\mathbb{C}^d$ can be arbitrarily approximated by a *tight* frame with vectors in $(\mathbb{Q}+i\mathbb{Q})^d$, and hence there are *many* tight frames for rational inner product spaces. We investigate the "minimal field" for which there is a tight frame with a given Gramian. We then consider the rational vector space given the cyclotomic field $\mathbb{Q}(\omega)$, with ω a primitive n-th root of unity. We give a simple formula for the unique inner product which makes the n-th roots $1, \omega, \omega^2, \ldots, \omega^{n-1}$ into a tight frame for $\mathbb{Q}(\omega)$. From this, we conclude that the associated "canonical coordinates" have many nice properties, e.g., multiplication in $\mathbb{Q}(\omega)$ corresponds to convolution, which makes them well suited to computation. Along the way, we give a detailed description of the space of $\mathbb{Q}$-linear dependencies between the n-th roots, which includes a cyclically invariant tight frame.

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1. Introduction

Let $\mathcal{H}$ be a d-dimensional real or complex inner product space. A finite sequence of vectors $(f_j)_{j\in J}$ in $\mathcal{H}$ is a **tight frame** for $\mathcal{H}$ if (for some $A > 0$)

$$f = \frac{1}{A}\sum_{j\in J}\langle f, f_j\rangle f_j, \qquad \forall f \in \mathcal{H}. \tag{1.1}$$

These generalisations of orthonormal bases have recently found many applications, e.g., in signal analysis [12], quantum information theory [16] and orthogonal polynomials of several variables [19]. One of the key motivations is that for inner product spaces with additional structure it may be possible for a tight frame to have certain desirable properties which it is impossible for a basis to have. In the infinite dimensional setting, this has been played out in the theories of wavelets and Gabor systems [3,13], to construct systems with good time–frequency localisation.

The theory of finite tight frames is still in its foundational stages [4]. There is an ongoing effort to construct tight frames with certain properties. Most notably, a set of d^2 equiangular lines in $\mathbb{C}^d$ [17], i.e., d^2 unit vectors (f_j) in $\mathbb{C}^d$ with

$$|\langle f_j, f_k\rangle| = \frac{1}{\sqrt{d+1}}, \qquad j \neq k.$$

Central to such constructions (Zauner's conjecture, the SIC problem, spherical 2-designs with the maximal number of vectors) is a description of a subfield of $\mathbb{C}$ in which the inner products lie.

The purpose of this paper is to investigate tight frames for inner product spaces where the field $\mathbb{F}$ is a subfield of $\mathbb{C}$, most notably the rationals $\mathbb{F} = \mathbb{Q}$. This is closely related to the above question of what is the smallest field that a unitary image of a given frame can lie in (so that symbolic calculations can be done). We motivate these questions, and our answers to them, by a careful consideration of the Mercedes-Benz frame (three equally spaced unit vectors in $\mathbb{R}^2$). Key results and observations include:

- Inner products on $\mathbb{Q}$-vector spaces may not be isomorphic (unlike those for $\mathbb{C}$ and $\mathbb{R}$). Nevertheless, a tight frame for a rational inner product space is still determined (up to unitary equivalence) by its Gramian.
- An $n \times n$ matrix Q with entries in a subfield $\mathbb{F} \subset \mathbb{C}$ is the Gramian of a tight frame of n vectors for a d-dimensional inner product space if and only if it is a positive scalar multiple of a rank d orthogonal projection matrix. Such a tight frame can be constructed
  1. In an $\mathbb{F}$-inner product space, by considering the columns of Q.
  2. In $\mathbb{E}^d$, with the Euclidean inner product, where $\mathbb{E}$ is possibly larger than $\mathbb{F}$, by considering the rows of Q.
- A tight frame for $\mathbb{C}^d$ can be arbitrarily approximated by one in $(\mathbb{Q} + i\mathbb{Q})^d$.