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Star complements and edge-connectivity in finite graphs

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ABSTRACT

Let G be a finite graph with H as a star complement for a non-zero eigenvalue μ. Let $\kappa'(G)$, $\delta(G)$ denote respectively the edge-connectivity and minimum degree of G. We show that $\kappa'(G)$ is controlled by $\delta(G)$ and $\kappa'(H)$. We describe the possibilities for a minimum cutset of G when $\mu \notin \{-1, 0\}$. For such μ, we establish a relation between $\kappa'(G)$ and the spectrum of H when G has a non-trivial minimum cutset $E \nsubseteq E(H)$.

1. Introduction

Let G be a finite simple graph with μ as an eigenvalue of multiplicity k. (Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0, 1)$-adjacency matrix A of G has dimension k.) A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is

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called a *star complement* for μ in G. We use the notation of [8], where the basic properties of star sets and star complements are established in Chapter 5.

If G has H as a star complement of order t, for an eigenvalue $\mu \notin \{-1, 0\}$, then either (a) G has order at most $\binom{t+1}{2}$, or (b) $\mu = 1$ and $G = K_2$ or $2K_2$ [2, Theorem 2.3]. Thus there are only finitely many graphs with a prescribed star complement H for some eigenvalue other than 0 or -1. In these circumstances, it is of interest to investigate properties of H that are reflected in G: connectedness is one such property, as observed in [11, Section 2]. It was shown in [13] that the vertex-connectivity $\kappa(G)$ is controlled by $\kappa(H)$ and the minimum degree $\delta(G)$. In particular, for each $k \in \mathbb{N}$, there exists a smallest non-negative integer $f(k)$ such that

$$\mu \notin \{-1, 0\},\ \kappa(H) \geq k,\ \delta(G) \geq f(k) \Rightarrow \kappa(G) \geq k.$$

Here we first establish an analogous result for edge-connectivity: for each $k \in \mathbb{N}$, there exists a smallest non-negative integer $g(k)$ such that

$$\mu \neq 0,\ \kappa'(H) \geq k,\ \delta(G) \geq g(k) \Rightarrow \kappa'(G) \geq k. \tag{1}$$

The arguments for $\kappa'(G)$ are quite different from those for $\kappa(G)$, and rely on a property of dominating sets. Moreover, whereas little is known about the function f, we find that $g(1) = 0$ and $g(k) = k$ for all $k > 1$. (It was shown in [13] that $k \leq f(k) \leq \frac{1}{2}(k-1)(k+2)$, while $f(1) = 0$, $f(2) = 2$, $f(3) = 3$, $f(4) = 5$, $f(5) = 7$ and $f(6) \geq 8$.)

We go on to investigate the nature of minimum cutsets of G when $\mu \notin \{-1, 0\}$. Following [9], we say that such a cutset E is *trivial* if E consists of the edges containing a vertex v (necessarily of degree $\delta(G)$). The interesting case is that in which G has a nontrivial minimum cutset E not in $E(H)$, for then we can find an upper bound for $\kappa'(G)$ in terms of the spectrum of H. We note some consequences in the case that H is regular and μ is not a main eigenvalue.

2. Preliminaries

We take $V(G) = \{1, \ldots, n\}$, and write $u \sim v$ to mean that vertices u and v are adjacent. The eigenvalues of G are denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, in non-increasing order. For $S \subseteq V(G)$, we write G_S for the subgraph induced by S, and $\Delta_S(u)$ for the S-neighbourhood $\{v \in S : v \sim u\}$. For the subgraph H of G we write $\Delta_H(u)$ for $\Delta_{V(H)}(u)$. An all-1 vector is denoted by $\mathbf{j}$, its length determined by context.

The following result, known as the Reconstruction Theorem, is fundamental to the theory of star complements.

Theorem 2.1. *(See [8, Theorem 5.1.7].) Let X be a set of k vertices in G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of G_X.*