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Nonsymmetric generic matrix equations

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Let $(A_i)_{0\leq i\leq k}$ be generic matrices over $\mathbb{Q}$, the field of rational numbers. Let $K=\mathbb{Q}(E)$, where E denotes the entries of the $(A_i)_i$, and let $\overline{K}$ be the algebraic closure of K. We show that the generic unilateral equation $A_kX^k+\cdots+A_1X+A_0=0_n$ has $\binom{nk}{n}$ solutions $X\in\mathcal{M}_n(\overline{K})$. Solving the previous equation is equivalent to solving a polynomial of degree kn, with Galois group S_{kn} over K. Let $(B_i)_{i\leq k}$ be fixed $n\times n$ matrices with entries in a field L. We show that, for a generic $C\in\mathcal{M}_n(L)$, a polynomial equation $g(B_1,\cdots,B_k,X)=C$ admits a finite fixed number of solutions and these solutions are simple. We study, when $n=2$, the generic non-unilateral equations $X^2+BXC+D=0_2$ and $X^2+BXB+C=0_2$. We consider the unilateral equation $X^k+C_{k-1}X^{k-1}+\cdots+C_1X+C_0=0_n$ when the $(C_i)_i$ are $n\times n$ generic commuting matrices; we show that every solution $X\in\mathcal{M}_n(\overline{K})$ commutes with the $(C_i)_i$. When $n=2$, we prove that the generic equation $A_1XA_2X+XA_3X+X^2A_4+A_5X+A_6=0_2$ admits 16 isolated solutions in $\mathcal{M}_2(\overline{K})$, that is, according to Bézout's theorem, the maximum for a quadratic 2×2 matrix equation.

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1. Introduction

S. Gelfand wrote in 2004 (cf. [10]): "The problem of solving quadratic equations for matrices seems to be absolutely classical. It would be natural that such a problem should have been at least formulated, or even solved, in the 19th century at the latest. Still, I asked many people about this problem, and they directed me to various sources, but nowhere could I find even a mention of this problem".

Let $n \in \mathbb{N}_{\geq 2}$. In the present paper, we deal with polynomial equations where the coefficients are generic $n\times n$ matrices and the unknown is an $n\times n$ matrix; the underlying field is assumed to have characteristic 0. Questions about generic matrices are solved in [1] and [2] or about formal matrices in [16]. More generally, C. Procesi described, in [20], properties of the algebra of generic matrices.

Let $\mathbb{Q}$ be the field of rational numbers. If M is an $n \times n$ matrix, then χ_M denotes its characteristic polynomial, $\sigma(M)$ its spectrum and $\mathrm{tr}(M)$ its trace.

Definition 1. (Cf. [6].) Let $\{a_{r;i,j} \mid 1 \leq i, j \leq n, 1 \leq r \leq k\}$ be independent commuting indeterminates over $\mathbb{Q}$; in other words, the $(a_{r;i,j})_{rij}$ are elements of a transcendental extension of $\mathbb{Q}$ and they are mutually transcendental over $\mathbb{Q}$. Then, when $r \in [\![1, k]\!]$, the $n \times n$ matrices $A_r = [a_{r;i,j}]$ are called *generic matrices* (over $\mathbb{Q}$); in the sequel, such matrices are assumed to be fixed. We consider the quotient field $K = \mathbb{Q}((a_{1;i,j})_{i,j}, \cdots, (a_{k;i,j})_{i,j})$ and its algebraic closure $\overline{K}$. Let f be a non-zero polynomial over K in $k + 1$ non-commuting indeterminates. We consider the so-called *generic* matrix equation:

$$f(A_1, \cdots, A_k, X) = 0_n \text{ in the unknown } X = [x_{i,j}] \in \mathcal{M}_n(\overline{K}). \tag{1}$$

i) Assume that the previous equation has a finite positive number of solutions. If the entries of each solution can be calculated by radicals over K, then we say that Eq. (1) is solvable, else we say that Eq. (1) is non-solvable.

ii) (Cf. [15].) A solution X_0 of Eq. (1) is called (geometrically) isolated if there is a neighborhood of X_0 that contains no other solution of the equation.

Let $k, n \geq 2$ and $(A_i)_{0\leq i\leq n}$ be $n\times n$ generic matrices; put $K = \mathbb{Q}((A_i)_i)$. In Section 2, we consider the unilateral equation of degree k in the unknown $X \in \mathcal{M}_n(\overline{K})$

$$A_kX^k + \cdots + A_1X + A_0 = 0_n. \tag{2}$$

Moreover, we study the nonsymmetric algebraic Riccati equation in $X \in \mathcal{M}_n(\overline{K})$

$$XAX + B_1X + XB_2 + C = 0_n, \tag{3}$$

where A, B_1, B_2, C are $n \times n$ generic matrices. We reduce the study of Eq. (3) to the following one

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