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Cartesian decomposition and numerical radius inequalities

Fuad Kittaneh [a,*], Mohammad Sal Moslehian [b], Takeaki Yamazaki [c]

[a] *Department of Mathematics, The University of Jordan, Amman, Jordan*
[b] *Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran*
[c] *Department of Electrical, Electronic and Computer Engineering, Toyo University, Kawagoe-Shi, Saitama, 350-8585 Japan*

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We show that if $T = H + iK$ is the Cartesian decomposition of $T \in \mathbb{B}(\mathscr{H})$, then for $\alpha, \beta \in \mathbb{R}$, $\sup_{\alpha^2+\beta^2=1} \|\alpha H + \beta K\| = w(T)$. We then apply it to prove that if $A, B, X \in \mathbb{B}(\mathscr{H})$ and $0 \le mI \le X$, then

$$\begin{aligned} m\big\|\mathrm{Re}(A) - \mathrm{Re}(B)\big\| &\le w\big(\mathrm{Re}(A)X - X\mathrm{Re}(B)\big) \\ &\le \frac{1}{2}\sup_{\theta\in\mathbb{R}}\big\|(AX - XB) + e^{i\theta}(XA - BX)\big\| \\ &\le \frac{\|AX - XB\| + \|XA - BX\|}{2}, \end{aligned}$$

where $\mathrm{Re}(T)$ denotes the real part of an operator T. A refinement of the triangle inequality is also shown.

* Corresponding author.
E-mail addresses: fkitt@ju.edu.jo (F. Kittaneh), moslehian@um.ac.ir (M.S. Moslehian), t-yamazaki@toyo.jp (T. Yamazaki).

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1. Introduction

Let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ and I stand for the identity operator. If $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the space $\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field and denote its identity by I_n. Any operator $T \in \mathbb{B}(\mathscr{H})$ can be represented as $T = H + iK$, the so-called Cartesian decomposition, where $H = \mathrm{Re}(T) = \frac{T+T^*}{2}$ and $K = \mathrm{Im}(T) = \frac{T-T^*}{2i}$ are called the real and imaginary parts of T. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive, denoted by $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$. For $p \geq 1$, the Schatten p-class, denoted by $\mathcal{C}_p$, is defined to be the two-sided ideal in $\mathbb{B}(\mathscr{H})$ of those compact operators A for which $\|A\|_p = \mathrm{tr}(|A|^p)^{1/p}$ is finite, where the symbol tr denotes the usual trace. This norm as well as the usual operator norm $\|\cdot\|$ are typical examples of unitarily invariant norms, i.e., a norm $|||\cdot|||$ defined on a two-sided ideal $\mathcal{C}_{|||\cdot|||}$ of $\mathbb{B}(\mathscr{H})$ satisfying $|||UAV||| = |||A|||$ for all $A \in \mathcal{C}_{|||\cdot|||}$ and all unitaries $U, V \in \mathbb{B}(\mathscr{H})$. The numerical radius of $A \in \mathbb{B}(\mathscr{H})$ is defined by

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathscr{H})$ such that for all $A \in \mathbb{B}(\mathscr{H})$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1.1}$$

If A is self-adjoint, then $w(A) = \|A\|$ and if $A^2 = 0$, then $w(A) = \frac{\|A\|}{2}$ (see e.g., [6] and [9]). Of course, $w(\cdot)$ is not unitarily invariant, rather it satisfies $w(U^*AU) = w(A)$ for all A and all unitary U in $\mathbb{B}(\mathscr{H})$, i.e., $w(\cdot)$ is weakly unitarily invariant.

Some interesting numerical radius inequalities improving inequalities (1.1) have been obtained by several mathematicians (see [1,5,6,13], and references therein). Several investigations on norm and numerical radius inequalities involving the Cartesian decomposition may be found in the literature, among them we would like to refer the reader to [4] and [7].

In this note, we show that if $T = H + iK$ is the Cartesian decomposition of $T \in \mathbb{B}(\mathscr{H})$, then for $\alpha, \beta \in \mathbb{R}$, $\sup_{\alpha^2+\beta^2=1} \|\alpha H + \beta K\| = w(T)$. We then apply it to find upper and lower bounds for $w(\mathrm{Re}(A)X - X\mathrm{Re}(B))$, where $A, B, X \in \mathbb{B}(\mathscr{H})$ and $0 \leq mI \leq X$. Furthermore, we present a refinement of the triangle inequality.

2. Results

We start this section with a result concerning the Cartesian decomposition.

Theorem 2.1. *Let $T = H + iK$ be the Cartesian decomposition of $T \in \mathbb{B}(\mathscr{H})$. Then for $\alpha, \beta \in \mathbb{R}$,*

$$\sup_{\alpha^2+\beta^2=1} \|\alpha H + \beta K\| = w(T). \tag{2.1}$$