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Primitive tensors and directed hypergraphs

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ABSTRACT

Primitivity is an important concept in the spectral theory of nonnegative matrices and tensors. It is well-known that an irreducible matrix is primitive if and only if the greatest common divisor of all the cycles in the associated directed graph is equal to 1. The main aim of this paper is to establish a similar result, i.e., we show that a nonnegative tensor is primitive if and only if the greatest common divisor of all the cycles in the associated directed hypergraph is equal to 1.

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1. Introduction

A real square matrix with non-negative elements is said to be nonnegative. The class of nonnegative matrices has been the subject of numerous investigations in matrix analysis and applications, see for instance [4,5]. One of the most important properties for a non-negative matrix is irreducibility.

Definition 1. (See [15].) An n-by-n nonnegative matrix $\mathbf{A} = (a_{i,j})$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ (or $\langle n \rangle$) such that $a_{i,j} = 0$ for all $i \in I$, $j \notin I$. If A is not reducible, then we call A irreducible.

In the literature, it has been shown there are several equivalent definitions of an irreducible matrix, see for instance [4,5]. Among them, it is interesting to use a directed graph to provide a geometric interpretation for the concept of irreducibility.

Definition 2. (See [5].) The associated directed graph, $G(\mathbf{A})$, of an n-by-n matrix $\mathbf{A}$, consists of n vertices p_i $(i = 1, \cdots, n)$ where an edge leads from p_i to p_j if and only if $a_{i,j} \neq 0$.

For simplicity, we denote the set of vertices and edges to be V and E respectively.

Definition 3. (See [5].) A directed graph G is strongly connected if for any ordered pair (p_i, p_j) of vertices of G, there exists a sequence of edges (a path) which leads from p_i to p_j.

The following theorem gives the relationship between irreducibility and strongly connectedness.

Theorem 1. *(See [5].) A nonnegative matrix* $\mathbf{A}$ *is irreducible if and only if its associated graph* $G(\mathbf{A})$ *is strongly connected.*

According to this theorem, the problem of determining a nonnegative matrix to be irreducible is equivalent to checking whether its associated graph is strongly connected or not. Besides irreducibility, primitivity is an important property of a nonnegative matrix.

Definition 4. (See [15].) A nonnegative matrix $\mathbf{A}$ is said to be primitive if it is irreducible and it has only one eigenvalue attained to its maximum modulus.

The following theorem gives an equivalent characterization of a primitive matrix.

Theorem 2. *(See [15].) If* $\mathbf{A}$ *is nonnegative, then* $\mathbf{A}$ *is primitive if and only if* $\mathbf{A}^m > 0$ *for some* $m \geq 1$.

The following theorem is a well-known result for the connection between a primitive matrix and its corresponding directed graph.