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Ranks of dense alternating sign matrices and their sign patterns

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ABSTRACT

In this paper, an explicit formula for the ranks of dense alternating sign matrices is obtained. The minimum rank and the maximum rank of the sign pattern of a dense alternating sign matrix are determined. Some related results and examples are also provided.

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1. Introduction

An *alternating sign matrix*, henceforth abbreviated ASM, is a square $(0, +1, -1)$-matrix without zero rows and columns, such that the $+1$s and -1s alternate in each row and column, beginning and ending with a $+1$, see [4]. Substantial interest in ASMs in the mathematics community originated from the alternating sign matrix conjecture of Mills et al. [16] in 1983 and has continued in several combinatorial directions. In [4] the authors initiated a study of the zero–nonzero patterns of $n \times n$ alternating sign matrices. In [3], the ASMs D_n which attain the maximum spectral radius over the set of $n \times n$ ASMs are identified, while in [10], inverses or generalized inverses of D_n are derived and the eigenvalues of D_n are considered. More recently, Brualdi and Kim have made even further progress on ASMs in [5–7].

A matrix is said to be *dense* (*row-dense, column-dense*) if there are no zeros between two nonzero entries for every line (row, column) of this matrix, see [12]. Some connections of dense alternating sign matrices with totally unimodular matrices, combined matrices, and generalized complementary basic matrices were explored in [12]. In particular, it was shown that every dense ASM is totally unimodular and that the combined matrix of every nonsingular dense ASM is an ASM.

An important part of the combinatorial matrix theory is the study of sign pattern matrices, which has been the focus of extensive research for the last 50 years [9,14]. A *sign pattern matrix*, or *sign pattern*, is a matrix whose entries are from the set $\{+, -, 0\}$. For a real matrix B, $\operatorname{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-, 0$). For a sign pattern matrix A, the *qualitative class* of A, denoted $Q(A)$, is defined as

$$Q(A) = \big\{B\colon\ B \text{ is a real matrix and } \operatorname{sgn}(B) = A\big\}.$$

A square sign pattern matrix A is said to be *sign nonsingular* if every matrix in $Q(A)$ is nonsingular.

The *minimum rank* of a sign pattern matrix A, denoted $\operatorname{mr}(A)$, is the minimum of the ranks of the real matrices in $Q(A)$. Determination of the minimum rank of a sign pattern matrix in general is a longstanding open problem in combinatorial matrix theory. Recently, there has been a significant number of papers concerning this topic, for example [1,2,11,13–15]. In particular, matrices realizing the minimum rank of a sign pattern have applications in the study of neural networks [11] and communication complexity [13].

The *maximum rank* of a sign pattern matrix A, denoted $\operatorname{MR}(A)$, is the maximum of the ranks of the real matrices in $Q(A)$. It should be clear that $\operatorname{MR}(A)$ is the maximum number of nonzero entries of A, no two of which are in the same row or in the same column. The maximum number of nonzero entries of A with no two of the nonzero entries in the same line (a row or column) is also known as the *term rank* of A [8,9].

The following version of Konig's Theorem [8] provides another description of the maximum rank.