

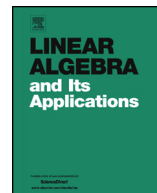


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Maximizing algebraic connectivity for certain families of graphs



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ABSTRACT

We investigate the bounds on algebraic connectivity of graphs subject to constraints on the number of edges, vertices, and topology. We show that the algebraic connectivity for any tree on n vertices and with maximum degree d is bounded above by $2(d-2)\frac{1}{n} + O(\frac{\ln n}{n^2})$. We then investigate upper bounds on algebraic connectivity for cubic graphs. We show that algebraic connectivity of a cubic graph of girth g is bounded above by $3 - 2^{3/2} \cos(\pi/\lfloor g/2 \rfloor)$, which is an improvement over the bound found by Nilli [34]. Finally, we propose several conjectures and open questions.

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1. Introduction

This paper is motivated by the following question: among all possible networks connecting n nodes, and subject to a specified resource or topology constraints, which one is the most effective at diffusing the flow of information? We are interested in the case where the network is undirected and all non-zero edges have the same weight.

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One of the simplest ways of modelling the information flow in a network is the linear consensus model, which is widely used in control theory [1]:

$$\frac{du_j}{dt} = \sum_{j \neq i} e_{ij}(u_i - u_j). \quad (1)$$

Here e_{ij} denote edge weights between nodes i, j and u_j is the “load” at node j ; the information flows from i to j in proportion to the load differential between the nodes; $e_{ij} = 1$ if i and j are joined by an edge and is zero otherwise. For large t the solution to (1) is given by $u(t) \sim \bar{u} + Ce^{-\lambda_2 t}$, where $\bar{u}$ is consensus (average) state and λ_2 is the second smallest eigenvalue of the graph Laplacian matrix $L = D - A$ where A is the adjacency matrix and D is the degree matrix (the smallest eigenvalue of L is zero and $\lambda_2 > 0$ if and only if the network is connected). The eigenvalue λ_2 is often called the *algebraic connectivity* of the graph [2], and roughly, the larger λ_2 , the faster u diffuses to its consensus state. In this sense, the “optimal” network is the one which maximizes the algebraic connectivity, subject to given constraints. This leads to the following question.

Question. Which graphs maximize the algebraic connectivity, given a set of constraints on the number of vertices, edges, maximum degree, and graph topology?

This and related questions arise in many diverse areas, including optimal network topologies [3]; scheduling and network coding [4]; experimental design [5,6], diffusion in small world networks [7,8], synchronization in complex networks [9], and ranking algorithms [10,11]. There is also a close link to expander graphs and Ramanujan graphs. These are graphs with “high” algebraic connectivity in some sense. See recent reviews [12,13] and references therein. A nice recent survey on algebraic connectivity is [14].

In general, the problem of finding the optimal graph given m edges and n vertices is known to be NP-complete [15]. Despite this fact, several simple heuristics exist that can be used to obtain a graph with reasonably large algebraic connectivity [16,17]. See also [18] for some results for almost-complete graphs, where m is close to $n(n-1)/2$. In [19,20], the question of optimizing algebraic connectivity with respect to graph diameter was studied.

In this paper we are concerned with the regime where the number of edges m grows in proportion to the number of vertices n , so that the graph is relatively sparse. In particular, a random Erdos–Renyai graph with $O(n)$ edges is well known to be disconnected with high probability as $n \rightarrow \infty$, so for such a graph, $\lambda_2 = 0$ almost surely [21,22], and as such, random graphs are not good optimizers in this regime. The smallest value for m for which the graph is connected is $m = n - 1$, in which case any connected graph is a tree (for disconnected graphs, $\lambda_2 = 0$ so we only consider connected case). Without a degree restriction, the star, which is a tree having a single root and $n - 1$ leaves (see Fig. 1(a)), is the unique optimizer of algebraic connectivity among all trees of n vertices, with $\lambda_2 = 1$ when $n \geq 3$ [23,14]. However, many trees of importance to applications

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