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On the inertia set of a signed graph with loops

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ABSTRACT

A signed graph is a pair (G, Σ), where $G = (V, E)$ is a graph (in which parallel edges and loops are permitted) with $V = \{1, \dots, n\}$ and $\Sigma \subseteq E$. The edges in Σ are called odd edges and the other edges of E even. By $S(G, \Sigma)$ we denote the set of all symmetric $n \times n$ real matrices $A = [a_{i,j}]$ such that if $a_{i,j} < 0$, then among the edges connecting i and j, there must be at least one even edge; if $a_{i,j} > 0$, then among the edges connecting i and j, there must be at least one odd edge; and if $a_{i,j} = 0$, then either there must be at least one odd edge and at least one even edge connecting i and j, or there are no edges connecting i and j. (Here we allow $i = j$.) For a symmetric real matrix A, the partial inertia of A is the pair (p, q), where p and q are the numbers of positive and negative eigenvalues of A, respectively. If (G, Σ) is a signed graph, we define the *inertia set* of (G, Σ) as the set of the partial inertias of all matrices $A \in S(G, \Sigma)$.

In this paper, we present formulas that allows us to obtain the minimal elements of the inertia set of (G, Σ) in case (G, Σ) has a 1-separation using the inertia sets of certain signed graphs associated with the 1-separation.

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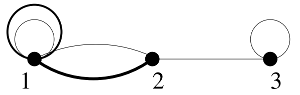

Fig. 1. Thick edges denote odd edges, while thin edges denote even edges.

0. Introduction

A signed graph is a pair (G, Σ), where $G = (V, E)$ is a graph (in which parallel edges and loops are permitted) with $V = \{1, \ldots, n\}$ and $\Sigma \subseteq E$. (We refer to [2] for the notions and concepts in graph theory.) The edges in Σ are called odd edges and the other edges of E even edges. By $S(G, \Sigma)$ we denote the set of all symmetric $n \times n$ real matrices $A = [a_{i,j}]$ such that

- if $a_{i,j} < 0$, then among the edges connecting i and j, there must be at least one even edge,
- if $a_{i,j} > 0$, then among the edges connecting i and j, there must be at least one odd edge, and
- if $a_{i,j} = 0$, then either there must be at least one odd edge and at least one even edge connecting i and j, or there are no edges connecting i and j.

Here we allow $i = j$, in which case loops might occur at vertex i. For example, the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

belongs to $S(G, \Sigma)$, where (G, Σ) is the signed graph shown in Fig. 1. For a symmetric real matrix A, the partial inertia of A, denoted by $\mathrm{pin}(A)$, is the pair (p, q), where p and q are the numbers of positive and negative eigenvalues of A, respectively. If (G, Σ) is a signed graph, we define the *inertia set* of (G, Σ) as the set $\{\mathrm{pin}(A) \mid A \in S(G, \Sigma)\}$; we denote the inertia set of (G, Σ) by $\mathcal{I}(G, \Sigma)$.

The *maximum rank* of a signed graph (G, Σ) is $\max\{\mathrm{rank}(A) \mid A \in S(G, \Sigma)\}$ and is denoted by $\mathrm{MR}(G, \Sigma)$. Observe that the maximum rank of a signed graph with loops need not equal the number of vertices of G. An example is the signed graph $(K_1, \emptyset)$, the signed graph consisting of one vertex and no edges. The maximum rank of this signed graph is 0, while its number of vertices is 1.

A *separation* of a graph $G = (V, E)$ is a pair (G_1, G_2) of subgraphs of G such that $G_1 \cup G_2 = G$ and $E(G_1) \cap E(G_2) = \emptyset$; its *order* is the cardinality of $V(G_1) \cap V(G_2)$. If the order of a separation is k, we also say that (G_1, G_2) is a k-separation. The notions of separations transfer without change to signed graphs.

Before presenting the formulas, we need introduce some other notation. Let $\mathcal{S}, \mathcal{R} \subseteq \mathbb{N}^2$; in this paper, we include 0 in the set $\mathbb{N}$. If for each $(p, q) \in \mathcal{S}$, there exists an $(r, s) \in \mathcal{R}$ such that $r \leq p$ and $s \leq q$, then we write $\mathcal{S} \leq \mathcal{R}$. Call a pair $(p, q) \in \mathcal{S}$ *minimal* if there