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On the generalized Clifford algebra of a monic polynomial

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In this paper we study the generalized Clifford algebra defined by Pappacena of a monic (with respect to the first variable) homogeneous polynomial $\Phi(Z, X_1, \ldots, X_n) = Z^d - \sum_{k=1}^{d} f_k(X_1, \ldots, X_n)Z^{d-k}$ of degree d in $n+1$ variables over some field F. We completely determine its structure in the following cases: $n = 2$ and $d = 3$ and either $\mathrm{char}(F) = 3$, $f_1 = 0$ and $f_2(X_1, X_2) = eX_1X_2$ for some $e \in F$, or $\mathrm{char}(F) \neq 3$, $f_1(X_1, X_2) = rX_2$ and $f_2(X_1, X_2) = eX_1X_2 + tX_2^2$ for some $r, t, e \in F$. Excluding a few exceptions, this algebra is an Azumaya algebra of rank nine whose center is the coordinate ring of an affine elliptic curve. We also discuss representations of arbitrary generalized Clifford algebras assuming the base field F is algebraically closed of characteristic zero.

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1. Introduction

Given a homogeneous polynomial form $f(X_1,\dots,X_n)$ in n variables of degree d over a field F, its Clifford algebra C_f is defined to be the quotient of the free associative F-algebra $F\langle x_1,\dots,x_n\rangle$ by the ideal generated by the elements $(a_1x_1+\dots+a_nx_n)^d-f(a_1,\dots,a_n)$ for all $a_1,\dots,a_n\in F$. This definition was introduced by Roby in [18]. It generalizes the classical notion of the Clifford algebra of a quadratic form. If the quadratic form is regular, its Clifford algebra is known to be a tensor product of quaternion algebras over F if the dimension is even or over a quadratic extension of F if the dimension is odd (see [15] or [13]).

The Clifford algebra is invariant under linear change of variables, and therefore it is used in studying the isometry classes of homogeneous polynomials. Another use of the Clifford algebra is in the study of central simple algebras. Given a central simple algebra A that contains an F-vector subspace V such that $v^d\in F$ for all $v\in V$, the subalgebra $F[V]$ of A is a representation of the Clifford algebra C_f where $f(X_1,\dots,X_n)=(v_1X_1+\dots+v_nX_n)^d$ for some fixed basis $v_1,\dots,v_n$ of V. Consequently, the representations of the Clifford algebra, and in particular the irreducible ones (which correspond to the simple images), are of importance.

The case of $n=2$ and $d=3$ was first considered by Heerema in [12]. Haile studied these algebras in a series of papers [6–9]. He showed that in characteristic not 2 or 3, if the binary cubic form f is nondegenerate, C_f is an Azumaya algebra of rank 9, whose center is isomorphic to the coordinate ring of the affine elliptic curve $s^2=r^3-27\Delta$ where Δ is the discriminant of f. Hence the simple homomorphic images of C_f are cyclic algebras of degree 3. Moreover, there is a one-to-one correspondence between the Galois orbits of points on that affine elliptic curve and the simple homomorphic images of C_f. The induced map from the group of F-rational points on the elliptic curve described above with the point at infinity as the unit element to the Brauer group of F is a group homomorphism whose image is the relative Brauer group of the function field of the curve C given by $Z^3-f(X,Y)=0$. He also proved that C_f is split if and only if the curve C has an F-rational point.

The irreducible representations of Clifford algebras were studied in papers such as [4,11,19]. It was shown in [11] that if f is nondegenerate and F is an infinite field, the dimension of any representation of C_f is divisible by the degree of f. In case F is algebraically closed of characteristic 0, Van den Bergh in [19] showed that the equivalence classes of dr-dimensional representations of C_f are in one-to-one correspondence with the isomorphism classes of certain vector bundles (later known as Ulrich bundles) of rank r on the hypersurface in $\mathbb{P}^n$ given by the equation $Z^d-f(X_1,\dots,X_n)=0$. Advanced results on the representations of Clifford algebras were obtained in [4] via the study of Ulrich bundles.

In [16], Pappacena generalized the notion of the Clifford algebra to the algebra associated to a monic polynomial (with respect to the first variable) over F of the form $\Phi(Z,X_1,\dots,X_n)=Z^d-\sum_{k=1}^d f_k(X_1,\dots,X_n)Z^{d-k}$ where each f_k is a homogeneous