Linear Algebra and its Applications 471 (2015) 241–263

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Construction of real skew-symmetric matrices from interlaced spectral data, and graph

Keivan Hassani Monfared [a,*], Sudipta Mallik [b]

[a] *Department of Mathematics, Western Illinois University, 476 Morgan Hall, 1 University Circle, Macomb, IL 61455, USA*
[b] *Department of Mathematics & Statistics, Northern Arizona University, 805 S. Osborne Dr. PO Box: 5717, Flagstaff, AZ 86011, USA*

A R T I C L E I N F O

Article history:
Received 10 April 2014
Accepted 1 January 2015
Available online 21 January 2015
Submitted by M. Tsatsomeros

MSC:
05C50
65F18

Keywords:
Skew-symmetric matrix
Graph
Tree
Structured inverse eigenvalue problem
The Duarte property
Cauchy interlacing inequalities
The Jacobian method

A B S T R A C T

A 1989 result of Duarte asserts that for a given tree T on n vertices, a fixed vertex i, and two sets of distinct real numbers L, M of sizes n and $n-1$, respectively, such that M strictly interlaces L, there is a real symmetric matrix A such that graph of A is T, eigenvalues of A are given by L, and eigenvalues of $A(i)$ are given by M. In 2013, a similar result for connected graphs was published by Hassani Monfared and Shader, using the Jacobian method. Analogues of these results are presented here for real skew-symmetric matrices whose graphs belong to a certain family of trees, and all of their supergraphs.

Published by Elsevier Inc.

* Corresponding author.
E-mail addresses: k1monfared@gmail.com (K. Hassani Monfared), Sudipta.Mallik@nau.edu (S. Mallik).

http://dx.doi.org/10.1016/j.laa.2015.01.001
0024-3795/Published by Elsevier Inc.

1. Introduction

Inverse eigenvalue problems (IEP's) have long been studied because of many applications that they have in various areas of science and engineering [1,2]. That is, to find a matrix in a certain family of matrices with prescribed eigenvalues, eigenvectors, or both. In particular, structured inverse eigenvalue problems (SIEP's) have received a lot of attention [3]. For example, one might be interested in finding matrices which have prescribed eigenvalues where the solution matrix has a certain zero-nonzero pattern. In this paper we study an SIEP which asks about the existence of a real skew-symmetric matrix with a specific zero-nonzero pattern where the eigenvalues of the matrix and the eigenvalues of a principal submatrix of it are prescribed and are distinct. We shall give a precise formulation of the problem (which we call the $\lambda - \mu$ skew-symmetric SIEP), and a solution when the structure of the matrix is defined by a family of trees and their supergraphs.

Cauchy interlacing inequalities [4] assert that the eigenvalues of a real symmetric matrix and those of a principal submatrix of it satisfy certain inequalities. Namely, if A is an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and B is an $(n-1) \times (n-1)$ principal submatrix of A with eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$. Then

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n. \tag{1.1}$$

Note that Cauchy interlacing inequalities hold for any Hermitian matrix A in general. The *spectrum* of a square matrix A, denoted by $\sigma(A)$, is the set of eigenvalues of A. For the preceding A and B, we say that $\sigma(B)$ *interlaces* $\sigma(A)$. If the inequalities in (1.1) are all strict, we say $\sigma(B)$ *strictly interlaces* $\sigma(A)$.

Here we introduce similar inequalities to Cauchy interlacing inequalities for skew-symmetric matrices. Since all the eigenvalues of any skew-symmetric matrix are purely imaginary numbers, we define an ordering on the imaginary axis of the complex plane. Let i denote the complex number $\sqrt{-1}$ and $\mathrm{i}\mathbb{R} = \{\mathrm{i}a : a \in \mathbb{R}\}$. For $a, b \in \mathrm{i}\mathbb{R}$ we say $a \leq b$ whenever $-\mathrm{i}a \leq -\mathrm{i}b$, and the equality holds if and only if $a = b$. Let $\mathcal{S} = \{a_1, \ldots, a_n\}$ be a subset of $\mathrm{i}\mathbb{R}$. $\mathcal{S}$ is said to be *presented in increasing order* if $a_1 \leq a_2 \leq \cdots \leq a_n$. Throughout this article we always present spectra of skew-symmetric matrices in increasing order. If $\mathcal{S} = \{a_1, \ldots, a_n\}$ is presented in increasing order, then a_1 is said to be the *smallest element* of $\mathcal{S}$, a_2 is said to be the *second smallest element* of $\mathcal{S}$, and so on.

Let $\mathcal{A} = \{\lambda_1, \ldots, \lambda_n\}$ and $\mathcal{B} = \{\mu_1, \ldots, \mu_{n-1}\}$ be subsets of $\mathrm{i}\mathbb{R}$, presented in increasing order. $\mathcal{B}$ is said to *interlace* $\mathcal{A}$ if $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$. Similarly $\mathcal{B}$ is said to *strictly interlace* $\mathcal{A}$ if $\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n$. Now Cauchy interlacing inequalities for skew-symmetric matrices can be stated as follows.

Theorem 1.1 *(Cauchy interlacing inequalities for skew-symmetric matrices). Let A be an $n \times n$ real skew-symmetric matrix and B be an $(n-1) \times (n-1)$ principal submatrix of A. Then $\sigma(B)$ interlaces $\sigma(A)$.*

Proof. Let A be an $n \times n$ real skew-symmetric matrix and B be an $(n-1) \times (n-1)$ principal submatrix of A. Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(B) = \{\mu_1, \ldots, \mu_{n-1}\}$.