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# On pseudo-inverses of matrices and their characteristic polynomials in supertropical algebra ☆

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The only invertible matrices in tropical algebra are diagonal matrices, permutation matrices and their products. However, the pseudo-inverse $A^\nabla$, defined as $\frac{1}{\det(A)}\operatorname{adj}(A)$, with $\det(A)$ being the tropical permanent (also called the tropical determinant) of a matrix $A$, inherits some classical algebraic properties and has some surprising new ones. Defining $B$ and $B'$ to be tropically similar if $B' = A^\nabla BA$, we examine the characteristic (max-)polynomials of tropically similar matrices as well as those of pseudo-inverses. Other miscellaneous results include a new proof of the identity for $\det(AB)$ and a connection to stabilization of the powers of definite matrices.



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## 1. Introduction

The **tropical max-plus semifield** is an ordered group $\mathcal{G}$ (usually the set of real numbers $\mathbb{R}$ or the set of rational numbers $\mathbb{Q}$), together with $-\infty$, denoted as $\mathbb{T} = \mathcal{G} \cup \{-\infty\}$, equipped with the operations $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$, denoted as $a + b$ and $ab$ respectively (see [1] and [14]). The unit element $1_{\mathbb{T}}$ is actually the element $0 \in \mathbb{Q}$. This arithmetic enables one to simplify non-linear questions by asking them in a (pseudo)-linear setting (see [12]), which can be applied to discrete mathematics, optimization, algebraic geometry and more, as has been well reviewed in [8,9,11,13,21] and [24]. In this max-plus language, we may also use notions of linear algebra to interpret combinatorial problems, such as eigenvectors being used to solve the Longest-Distance problem (see [4]).

The intention of this paper is to use an analogous concept of the inverse matrix by passing to a wider structure called the **supertropical semiring**, equipped with the ghost ideal $G = \mathcal{G}^\nu$, as established and studied by Izhakian and Rowen in [16] and [17]. The use of the term "pseudo" in this paper is the same as "quasi" throughout the work of Izhakian and Rowen.

We denote the "standard" supertropical semiring as

$$R = T \cup G \cup \{-\infty\},$$

where $T = \mathcal{G}$, which contains the so called tangible elements of the structure and where $\forall a \in T$ we have $a^\nu \in G$ are the ghost elements of the structure, as defined in [16]. So $G$ inherits the order of $\mathcal{G}$. This enables us to distinguish between a maximal element $a$ that is attained only once in a sum, i.e. $a \in T$ which is invertible, and a maximum that is being attained at least twice, i.e. $a + a = a^\nu \in G$, which is not invertible. Note that $\nu$ projects the standard supertropical semiring onto $G$, which can be identified with the usual tropical structure.

In this new supertropical sense, we use the following order relation to describe two elements that are equal up to some ghost supplement:

**Definition 1.1.** Let $a, b$ be any two elements in $R$. We say that $a$ **ghost surpasses** $b$, denoted $a \models_{gs} b$, if $a = b + \textit{ghost}$, i.e. $a = b$ or $a \in G$ with $a^\nu \geq b^\nu$. We say $a$ is $\nu$-equivalent to $b$, denoted by $a \cong_\nu b$, if $a^\nu = b^\nu$. That is, when we use $\nu$ to project from $R$ to $G$, identified with the tropical structure, $\nu$-equivalence becomes equality.

For matrices $A = (a_{ij}), B = (b_{ij}) \in M_{n\times m}(R)$ (and in particular for vectors) $A \models_{gs} B$ means $a_{ij} \models_{gs} b_{ij}$ $\forall i = 1, \dots, n$ and $j = 1, \dots, m$.

For polynomials

$$f(x) = \sum_{i=1}^{n} a_i x^i, \qquad g(x) = \sum_{i=1}^{n} b_i x^i \in R[x],$$

we say that $f(x) \models_{gs} g(x)$, also denoted as $f \models_{gs} g$, when $a_i \models_{gs} b_i$ $\forall i$.