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A Bezoutian approach to orthogonal decompositions of trace forms or integral trace forms of some classical polynomials

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As Helmke and Fuhrmann pointed out, Bezoutian approaches have been considered to be fruitful for the study of trace forms. In this article, we study orthogonal decompositions of trace forms or integral trace forms of some classical polynomials via Bezoutians. In Section 3, we give another proof of a theorem of Feit about orthogonal decompositions of trace forms of generalized Laguerre polynomials. In Section 4, we consider integral trace forms of certain irreducible trinomials and give their orthogonal decompositions explicitly. Then, in Section 5, we obtain their canonical forms over $\mathbb{Z}_p$ the ring of p-adic integers.

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1. Introduction

Let F be a field, let $f_1(x)$, $f_2(x)$ be polynomials in $F[x]$, and let $n \in \mathbb{Z}$ be greater than or equal to $\max\{\deg f_1, \deg f_2\}$. Then we put

$$\begin{aligned} B_n(f_1, f_2) &:= \frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x - y} \\ &= \sum_{i,j=1}^{n} \alpha_{ij} x^{i-1} y^{j-1} \in F[x, y], \\ M_n(f_1, f_2) &:= (\alpha_{ij})_{1 \le i,j \le n}. \end{aligned}$$

The matrix $M_n(f_1, f_2)$ is called the Bezoutian of f_1 and f_2. As is well-known, the Bezoutian $M_n(f_1, f_2)$ is a symmetric matrix and hence defines a symmetric F-bilinear form on F^n. We call this symmetric F-bilinear form the *Bezoutian form* of f_1 and f_2. In the article [8, p. 1042], Helmke and Fuhrmann pointed out the fruitfulness of Bezoutian approaches to the study of trace forms.

Let $f \in F[x]$ be a separable polynomial over F and set $K = F[x]/(f)$. It is well-known that the map

$$\mathrm{Tr}_{K/F} : K \times K \to F; \quad (x, y) \to \mathrm{trace}_{K/F}(xy)$$

defines a symmetric F-bilinear form on K, which we call the trace form of f (or K) and we denote by Tr_f the symmetric F-bilinear form space $(K, \mathrm{Tr}_{K/F})$. Suppose f is irreducible over $F = \mathbb{Q}$ the field of rational numbers and let O_K be the ring of integers of the number field K. Then, since $\mathrm{trace}_{K/\mathbb{Q}}(\alpha) \in O_K \cap \mathbb{Q} = \mathbb{Z}$ ($\alpha \in O_K$), the restriction of $\mathrm{Tr}_{K/\mathbb{Q}}$ to O_K defines a symmetric $\mathbb{Z}$-bilinear form:

$$\mathrm{Tr}_{K/\mathbb{Q}}|_{O_K} : O_K \times O_K \to \mathbb{Z}; \quad (x, y) \to \mathrm{trace}_{K/\mathbb{Q}}(xy),$$

which we call the integral trace form of f (or K) and we also denote by tr_f the symmetric $\mathbb{Z}$-bilinear form space O_K. Let us denote by $\mathrm{tr}_{f,p}$ the symmetric $\mathbb{Z}_p$-bilinear form space $\mathbb{Z}_p \otimes O_K$, the scalar extension of the symmetric $\mathbb{Z}$-bilinear form space O_K to the ring of p-adic integers $\mathbb{Z}_p$. The basic idea of this article is to consider (integral) trace forms to be Bezoutian forms, which will be reviewed in the next section. In [13], as a first step, we applied this idea to the case of cyclotomic fields $\mathbb{Q}(\zeta_n)$ and obtained orthogonal decompositions of symmetric $\mathbb{Z}$-bilinear form spaces $\mathrm{tr}_{\mathbb{Q}(\zeta_n)}$ explicitly. In this article, we attempt to expand this idea to a wider range of interesting cases.

In Section 3, we consider trace forms of generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ and Hermite polynomials $H_n(x)$. For generalized Laguerre polynomials, we give another proof of a theorem of Feit [6, Theorem 2.2, Theorem 5.1] about orthogonal decompositions of symmetric $\mathbb{Q}$-bilinear form spaces $\mathrm{Tr}_{L_n^{(\alpha)}}$. Using this theorem and a theorem of Serre [17, Theorem 1], Feit showed, for any $n \equiv 3 \pmod 4$, the splitting field of the generalized