Linear Algebra and its Applications 471 (2015) 346–352

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A matrix identity and its applications ☆

Jun Liao *, Heguo Liu, Minfeng Shao, Xingzhong Xu

School of Mathematics and Statistics, Hubei University, Wuhan, 430062, PR China

ARTICLE INFO

Article history:
Received 19 September 2014
Accepted 24 December 2014
Available online 28 January 2015
Submitted by X. Zhan

MSC:
15A24
15A21

Keywords:
Characteristic polynomial
Minimal polynomial
Rational canonical form

ABSTRACT

We first prove a matrix identity concerning the blocks of generalized Jordan blocks and then give applications to some invariants of matrices. As a consequence, we reprove the well known fact that for an eigenvalue λ, its algebraic multiplicity is greater than or equal to its geometric multiplicity.

1. Introduction

Let F be a field. A generalized Jordan block over F is a matrix of the form

$$\begin{bmatrix} C(p) & 0 & \cdots & 0 \\ N & C(p) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & N & C(p) \end{bmatrix}$$

☆ Supported by the National Natural Science Foundation of China (11371124 and 11401186).

* Corresponding author.
E-mail addresses: jliao@hubu.edu.cn (J. Liao), ghliu@hubu.edu.cn (H. Liu), smf2009@163.com (M. Shao), xuxingzhong407@126.com (X. Xu).

http://dx.doi.org/10.1016/j.laa.2014.12.032
0024-3795/

where $C(p)$ is the companion matrix of an irreducible polynomial $p = p(\lambda)$ over F of degree r, and N is an $r \times r$ matrix all of whose entries are zero except at the $(1, r)$ position, where the entry is 1. If there are e blocks on the diagonal, we denote it by $J(p^e)$. It is noteworthy that if F is algebraically closed then the generalized Jordan block becomes the usual Jordan block. The generalized Jordan form theorem states that every square matrix is similar to a direct sum of generalized Jordan blocks.

Theorem. *(See [1, Chap. 21, Theorem 5.5].) Let A be an $n \times n$ matrix over a field F, and $m_A = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s}$ be the factorization of the minimal polynomial of A into distinct monic irreducible factors. Then there exists an invertible matrix P such that $P^{-1}AP = J_1 \oplus J_2 \oplus \cdots \oplus J_s$, where $J_i = J(p_i^{e_{i1}}) \oplus J(p_i^{e_{i2}}) \oplus \cdots \oplus J(p_i^{e_{i\gamma_i}})$ with $e_{i\gamma_i} = \beta_i$.*

Note that if $p_i(\lambda) = \lambda - \xi_i$, then the generalized Jordan block $J(p_i^e)$ reduces to an elementary Jordan block. In particular, if F is an algebraically closed field, then the above canonical form is the well known Jordan normal form.

In this short note, we first give a matrix identity concerning the companion matrix $C(p)$ and N, which appear in a generalized Jordan block, and then we use it to describe some invariants of matrices as applications. Recall that for an eigenvalue λ, its algebraic multiplicity is the multiplicity of λ as a root of the characteristic polynomial. Its geometric multiplicity is the maximal number of linearly independent eigenvectors corresponding to it, see [2]. For an eigenvalue λ, it is well known that its algebraic multiplicity is greater than or equal to its geometric multiplicity. We will reprove it as a consequence of Corollary 4.

2. A matrix identity

In this section, we assume R is a commutative ring with identity 1. Let C be the companion matrix of $p(\lambda) = \lambda^r + a_{r-1}\lambda^{r-1} + \cdots + a_0$ in $R[\lambda]$, and let N be the $r \times r$ matrix all of whose entries are zero except $(1, r)$ entry is 1. That is

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \ddots & 0 & -a_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -a_{r-1} \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We first prove the following identity.

Theorem 1. *Let*

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \ddots & 0 & -a_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -a_{r-1} \end{bmatrix}, \qquad N = N_r = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{r \times r}$$